Notes on Complex Analysis

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May 15, 2025

Source: `https://github.com/slipperking/complex-analysis`

Contents

1 Prerequisites

1.1 Topological Preliminaries

The following definitions are subject to the assumption where the topological space is defined to be $X = \mathbb{C}^n$. This is satisfactory to the main purpose of our proceeding passage, but it is noteworthy that it can be generalized to more abstract sets (which will be discussed in later sections, especially in Section 3.2.1).

Definition 1.1.1 *(Open Ball)*: Let $B^n(a, r) \subset \mathbb{C}^n$ denote the n-dimensional *ball* with radius r centered at $a = (a_1, a_2, ..., a_n) \in \mathbb{C}^n$, defined by

$$B^n(a, r) = \left\{ (z_1, z_2, ..., z_n) \in \mathbb{C}^n \mid \sum_{j=1}^{n} \left|z_j - a_j\right|^2 < r^2 \right\}.$$

Definition 1.1.2 *(Open and Closed Sets)*: A set $U \subseteq \mathbb{C}^n$ is *open* if for every $z \in U$, there exists an *open ball* centered at z that is fully contained in U. A set $F \subseteq \mathbb{C}^n$ is *closed* if its complement, $\mathbb{C}^n \setminus F$, is open.

(Note that these definitions hold only for the standard topology on $\mathbb{C}^n$ induced by the Euclidean metric. For now, this is unimportant, but will be explained later in Section 3.2.1.)

Definition 1.1.3 *(Accumulation Point)*: A point $z \in \mathbb{C}^n$ is an *accumulation point* of X if for any open set U containing z, $(U \setminus \{z\}) \cap X \neq \varnothing$

Theorem 1.1.1: A set $X \subseteq \mathbb{C}^n$ is closed iff X contains all of its accumulation points.

Proof:

1 We first prove that closedness implies the inclusion of accumulation points. Let X be a closed set, and assume that some accumulation point z_0 of X satisfies $z_0 \notin X$. Since X is closed, $\mathbb{C}^n \setminus X$ is open, and there exists an open ball $B^n(z_0, \varepsilon)$ that is fully contained in $\mathbb{C}^n \setminus X$. However, this contradicts the definition of accumulation point, since $B^n(z_0, \varepsilon) \setminus \{z_0\}$ does not intersect with X.

2 Assume X is an arbitrary set which includes all its accumulation points. We will show that $\mathbb{C}^n \setminus X$ is open, which implies that X is closed. Let $z_0 \in \mathbb{C}^n \setminus X$. Since z_0 is not an accumulation point of X (as otherwise $z_0 \in X$), there exists an open ball $B^n(z_0, \varepsilon)$ such that $(B^n(z_0, \varepsilon) \setminus \{z_0\}) \cap X = \varnothing$. Then, $B^n(z_0, \varepsilon) \subset \mathbb{C}^n \setminus X$, and hence $\mathbb{C}^n \setminus X$ is open. $\square$

Definition 1.1.4 *(Closure)*: For a set $X \subseteq \mathbb{C}^n$, define the *closure* of X, or $\overline{X}$ to be the intersection of all closed sets containing X. In other words, it is the union of X and its accumulation points.

Because the accumulation points of X all lie in $\overline{X}$, the closure of X is a closed set. Moreover, X is closed iff $X = \overline{X}$.

Definition 1.1.5 *(Interior)*: For a set $X \subseteq \mathbb{C}^n$, the *interior* of X, denoted $\overset{\circ}{X}$, is the union of all open sets contained in X, or the set of points $z \in \mathbb{C}^n$ such that there exists an open neighborhood of z that is fully contained in X.

Definition 1.1.6 *(Compact Set)*: A set $X \subseteq \mathbb{C}^n$ is compact iff X is closed and bounded.

Definition 1.1.7 *(Set Covering)*: A cover $\mathcal{C}$ of a set X is a collection of sets $\{U_n\}$ such that

$$\bigcup_{n \in \mathbb{N}} U_n \supseteq X.$$

A cover is *open* if every set in the collection is open.

Theorem 1.1.2 *(Bolzano–Weierstrass)*: Every infinite subset A of a compact set $X \subseteq \mathbb{C}^n$ has an accumulation point in X.

Proof: Since X is bounded, there exists a closed cube $Q \subset \mathbb{C}^n$ such that $A \subseteq X \subset Q$.

Bisect $Q_0 = Q$ into 2^{2n} congruent sub-cubes. Since A is infinite and the sub-cubes are finite in number, at least one of the sub-cubes contains infinitely many points of A, and choose one to be Q_1.

Bisect Q_1 into 2^{2n} sub-cubes, and choose a sub-cube $Q_2 \subset Q_1$ that contains infinitely many points of A. We then obtain the recursive sequence

$$Q_0 \supset Q_1 \supset Q_2 \supset \cdots$$

Because the side lengths shrink to zero and the cubes are nested, the intersection

$$\bigcap_{k=0}^{\infty} Q_k$$

consists of exactly one point. Call this point $z_\infty \in \mathbb{C}^n$.

For each k, Q_k contains infinitely many points of A. Because the side length of Q_k tends to zero, for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall k \geq N$, $Q_k \subset B^n(z_\infty, \varepsilon)$.

Then, $B^n(z_\infty, \varepsilon)$ also contains infinitely many points of A. Therefore, z_∞ is an accumulation point of A.

We now show that $z_\infty \in X$. Suppose for contradiction that $z_\infty \notin X$. Since X is closed, $\mathbb{C}^n \setminus X$ is open, and $\exists \delta > 0$ such that

$$B^n(z_\infty, \delta) \subset \mathbb{C}^n \setminus X.$$

But then, for sufficiently large k, we have $Q_k \subset B^n(z_\infty, \delta)$, and hence $Q_k \cap X = \varnothing$. This contradicts the construction of Q_k, which ensures that Q_k contains infinitely many points of $A \subset X$. $\square$

Theorem 1.1.3 *(Heine–Borel)*: A set $X \subseteq \mathbb{C}^n$ is compact iff every open cover has a finite subcover.

Proof: We will first show that any set satisfying the condition is compact.

First we will show that X is bounded. Suppose that $\forall R > 0$, $\exists z \in X$ where $\|z\| > R$. Consider the collection of open sets

$$\mathcal{U} = \{B^n(0, k) \mid k \in \mathbb{N}\}.$$

$\mathcal{U}$ forms an open cover of X. Then by the assumption, there exists a finite subcover in $\mathcal{U}$, namely $\{B^n(0, k_1), ..., B^n(0, k_m)\}$ which covers X. Then,

$$X \subseteq \bigcup_{i=1}^{m} B^n(0, k_i) = B^n(0, \max\{k_1, ..., k_m\}).$$

By contradiction, X must be bounded.

X must also be a closed set. For the sake of contradiction, assume that there exists a point $z_0 \in \overline{X} \setminus X$. Since $z_0 \notin X$, the following open collection of sets covers X:

$$\mathcal{U} = \left\{\mathbb{C}^n \setminus \overline{B^n\left(z_0, \frac{1}{k}\right)} \mid \forall k \in \mathbb{N}\right\}.$$

There then exists a finite subcover $\mathcal{C} = \left\{\mathbb{C}^n \setminus \overline{B^n\left(z_0, \frac{1}{k_j}\right)} \mid j \in \mathbb{N}_{\leq m}\right\}$. Then,

$$X \subseteq \mathbb{C}^n \setminus \overline{B^n\left(z_0, \frac{1}{\max\{k_1, ..., k_m\}}\right)},$$

and that $X \cap \overline{B^n\left(z_0, \frac{1}{\max\{k_1, ..., k_m\}}\right)} = \varnothing$. However, by the definition of the accumulation point, every open neighborhood of the accumulation point must intersect X. Therefore, by contradiction, X is closed.

We then prove the converse. By the assumption that X is bounded, $\exists R > 0$ such that X is contained within the closed cube

$$Q = \left\{z \in \mathbb{C}^n \mid \max_{j \in \mathbb{N}_{\leq n}} |\mathfrak{Re}(z_j)| \leq R \land \max_{j \in \mathbb{N}_{\leq n}} |\mathfrak{Im}(z_j)| \leq R\right\}.$$

Assume that there exists an infinite open cover $\mathcal{U}$ of X without finite subcovering.

Bisect $Q_0 = Q$ into 2^{2n} sub-cubes (for real and complex parts), choose Q_1 such that $Q_1 \cup X$ has no finite subcover of $\mathcal{U}$.

Under the previous assumptions, this is possible since if every sub-cube $\cap$ X had finite subcovering, then $Q_0 \cap X = X$ would have finite subcovering.

Similarly, choose Q_2 by bisecting Q_1 similarly, and recursively obtain a sequence of cubes:

$$Q_0 \supset Q_1 \supset Q_2 \supset \cdots$$

Since the side length of each cube tends to 0, $\bigcap_{j=0}^{\infty} Q_j$ consists of a single point $z_\infty \in \mathbb{C}^n$. Since $\mathcal{U}$ covers X, $\exists U \in \mathcal{U}$ such that $z_\infty \in U$. Since U is open, $\exists \varepsilon > 0$ such that $B^n(z_\infty, \varepsilon) \subseteq U$. $\exists N \in \mathbb{N}$ such that $\forall k > N$, $Q_k \subset B^n(z_\infty, \varepsilon)$. Then taking the intersection with X on both sides,

$$Q_k \cap X \subseteq B^n(z_\infty, \varepsilon) \cap X \subseteq U.$$

This contradicts the assumption that for every k, $Q_k \cap X$ has no finite subcovering, since $\{U\} \subset \mathcal{U}$ clearly covers $Q_k \cap X$, as it is a single open set that covers a nonempty subset. Therefore by contradiction, every open cover has finite subcovering. $\square$

Definition 1.1.8 *(Support of a Function)*: For a set X and a function $f : X \to \mathbb{C}$, the *support*, denoted by $\operatorname{supp}(f) = \overline{\{z \in X \mid f(z) \neq 0\}}$, is the closure of the set for which f is nonzero.

Remark: A notable classification of functions comes from the compactness of support—more specifically, its boundedness. Compactly supported functions in C^∞ are commonly referred to as *bump functions* (see Section 3.2.1).

1.2 Calculus

Since traditional complex analysis is the theory of calculus on complex functions, it is only natural that generalizations are made on classical formulas in calculus for complex functions.

It is well known that a function $f : (a, b) \to \mathbb{R}$ is differentiable at a point $x \in (a, b)$ if the limit

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

exists, and the value of this limit is the derivative of $f(x)$, denoted by $f'(x)$ or $\frac{\mathrm{d}f}{\mathrm{d}x}$. The value $\mathrm{d}f = f'(x)\,\mathrm{d}x$ is the differential of $f(x)$. Partition $[a, b]$ into $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ such that the length of the intervals $[x_i, x_{i-1}]$ vanishes (we let the norm of the partition, or the size of the largest interval, tend to zero) as $n \to \infty$. If for any such partition, the sum

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

tends to the same value $\forall \xi_i \in [x_{i-1}, x_i]$ (as the length of the largest partition approaches 0), then the function can be roughly said to be integrable over $[a, b]$. The full details of Riemann integrability are simplified by the use of Darboux sums and will not be discussed here. The value of this sum is denoted by

$$\int_a^b f(x)\,\mathrm{d}x.$$

We will attempt to avoid notions involving Lebesgue integration. However, it is important to note that every Riemann integrable function is also Lebesgue integrable, and the two integrals are equal. Therefore, we will use Lebesgue integral theorems (where the resultant integral is Riemann integrable) when necessary without further mention of the Lebesgue integral itself.

The following theorems are the fundamental results of classical calculus:

Theorem 1.2.1 *(Fundamental Theorem of Calculus, Differential Form)*: Let $f(x)$ be a function continuous over $[a, b]$. For $x \in [a, b]$, define

$$\Phi(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Then $\Phi(x)$ is differentiable over $[a, b]$, $\Phi'(x) = f(x)$, and $\mathrm{d}\Phi(x) = f(x)\,\mathrm{d}x$.

Theorem 1.2.2 *(Fundamental Theorem of Calculus, Integral Form)*: Let $\Phi(x)$ be a function differentiable over $[a, b]$. Let $f(x) = \Phi'(x)$ over $[a, b]$. Then,

$$\int_a^x f(t)\,\mathrm{d}t = \Phi(x) - \Phi(a).$$

The two forms of the theorem show that differentiation and integration are inverse operations to each other. Operations performed for differentiating oftentimes have a corresponding inverse operation that can be done for integrating. For instance,

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) \pm g(x)) = f'(x) \pm g'(x)$$

corresponds to

$$\int (f(x) \pm g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

corresponds to

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x,$$

and

$$\frac{\mathrm{d}f(g(x))}{\mathrm{d}x} = \frac{\mathrm{d}f(g)}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$$

corresponds to

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u.$$

Another correspondence is the Mean Value Theorem:

Theorem 1.2.3 *(Mean Value Theorem, Differential Form)*: If $f(x)$ is differentiable over $[a, b]$, then $\exists c \in [a, b]$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Theorem 1.2.4 *(Mean Value Theorem, Integral Form)*: If $f(x)$ is continuous over $[a, b]$, then $\exists \xi \in [a, b]$ such that

$$\int_a^b f(x)\,\mathrm{d}x = f(\xi)(b - a).$$

A curve is a one-dimensional manifold embedded within a higher dimensional space. They can be parameterized with a vector $\mathbf{F}(t) = (P(t), Q(t), R(t))$ of one parameter. In the complex plane, a curve is a complex-valued function $\gamma(t)$ for a real parameter $\alpha \leq t \leq \beta$. A curve is *closed* if $\gamma(\alpha) = \gamma(\beta)$. It is *smooth* if it is continuously differentiable, and its direction is defined to be the direction as t increases. If it is smooth everywhere except at a finite number of points, it is *piecewise smooth*. If it is of finite length, then the curve is said to be *rectifiable*. Piecewise smooth curves are rectifiable. A curve is *simple* if it is simple (non-self-intersecting), or if $\gamma(t_1) = \gamma(t_2)$ implies that $t_1 = t_2$. A simple closed curve is also called a *Jordan curve*.

Theorem 1.2.5 *(Jordan Curve Theorem)*: Let γ be a Jordan curve in $\mathbb{R}^2$. Then the set $\mathbb{R}^2 \setminus \gamma$ consists of exactly two connected subsets. One of them is the interior, denoted by $\mathrm{int}(\gamma)$, and is a bounded set, while the other is the exterior, denoted by $\text{ext}(\gamma)$, which is unbounded. Both of the two sets share the common boundary γ.

The theorem above seems trivial, but its rigorous proof in topology is extremely complex. The theorem itself can also be stated on $\mathbb{C}$ instead of $\mathbb{R}^2$. For a region U, the boundary is denoted ∂U. If the region bounded by any closed curve in U also lies in U, then it is a *simply connected* region. A connected region that is not simply connected is multiply connected. A region bound by 2 non-intersecting Jordan curves is doubly connected, and a region bound by n non-intersecting Jordan curves is traditionally known as n-connected. Lastly, any closed curve can degenerate to a single point or slit.

Generalizations of the differential and integral exist for multivariate functions. The partial differentials of $f(x, y, z)$, $\frac{\partial f}{\partial x}\,\mathrm{d}x$, $\frac{\partial f}{\partial y}\,\mathrm{d}y$, and $\frac{\partial f}{\partial z}\,\mathrm{d}z$ sum up to form the total differential, denoted by $\mathrm{d}f$. An important result in multivariable calculus allows the calculation of the derivatives of a definite integral with respect to its parameter.

Theorem 1.2.6 *(Leibniz Integral Rule)*: Let $f(x, u)$ be continuous on $a \leq x \leq b$, $c \leq u \leq d$, and suppose $a \leq \alpha(u), \beta(u) \leq b$ are differentiable functions of $c \leq u \leq d$. If f is continuously differentiable with respect to u, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\left(\int_{\alpha(u)}^{\beta(u)} f(x,u)\,\mathrm{d}x\right) &= \int_{\alpha(u)}^{\beta(u)} \frac{\partial f}{\partial u}(x,u)\,\mathrm{d}x \\ &\quad + \frac{\mathrm{d}\beta}{\mathrm{d}u} f(\beta(u),u) - \frac{\mathrm{d}\alpha}{\mathrm{d}u} f(\alpha(u),u).\end{aligned}$$

Four main classical theorems exist, relating a function and its line integral in 2 and 3 dimensions, line and surface (or area) integrals in 2 and 3 dimensions, and the surface and volume integrals in 3 dimensions:

Theorem 1.2.7 *(Gradient Theorem)*: Let γ be an oriented smooth curve in $\mathbb{R}^3$ with boundary points a and b. Then if $f \in C^1(\gamma)$

$$f|_{\partial\gamma} = f(b) - f(a) = \int_\gamma \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z.$$

Theorem 1.2.8 *(Green's Theorem)*: Let U be a positively oriented, multiply connected subset of $\mathbb{R}^2$ with a piecewise smooth oriented boundary ∂U. Suppose that $P(x, y), Q(x, y) \in C^1\left(\overline{U}\right)$. Then,

$$\oint_{\partial U} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y.$$

Theorem 1.2.9 *(Stokes' Theorem)*: Suppose that $S \subset \mathbb{R}^3$ is a positively oriented surface with a positively oriented, piecewise smooth boundary curve ∂S. Suppose that $P(x,y,z), Q(x,y,z), R(x,y,z) \in C^1\left(\overline{S}\right)$. Then,

$$\begin{aligned}&\oint_{\partial S} P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z \\ = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y\,\mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)&\mathrm{d}z\,\mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Theorem 1.2.10 *(Gauss' Theorem)*: Suppose that $V \subset \mathbb{R}^3$ is a positively oriented region with a positively oriented, piecewise smooth boundary surface ∂V. Suppose that $P(x,y,z), Q(x,y,z), R(x,y,z) \in C^1\left(\overline{V}\right)$. Then,

$$\oiint_{\partial V} P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}z\,\mathrm{d}x + R\,\mathrm{d}x\,\mathrm{d}y = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

In 3-dimensional $\mathbb{R}^3$ space, define a scalar valued function to be a 0-form, a linear combination of $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ to be a 1-form, and a linear combination of $\mathrm{d}y \wedge \mathrm{d}z$, $\mathrm{d}z \wedge \mathrm{d}x$, and $\mathrm{d}x \wedge \mathrm{d}y$ to be a 2-form, and $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ to be a 3-form, where $\wedge$ denotes an anti-commutative and associative product, where for any two differential forms ω_1 and ω_2

$$\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1.$$

Then consequently, for any differential form ω,

$$\omega \wedge \omega = 0.$$

We can generalize the operator d to increase the degree of a differential form. For instance,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z,$$

which is the definition of the total differential. For a 1-form in 3-dimensional space, $\omega_1 = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$, we can define the exterior derivative in a similar way:

$$\begin{aligned}
\mathrm{d}\omega_1 &= \mathrm{d}P \wedge \mathrm{d}x + \mathrm{d}Q \wedge \mathrm{d}y + \mathrm{d}R \wedge \mathrm{d}z \\
&= \left(\frac{\partial P}{\partial x}\,\mathrm{d}x + \frac{\partial P}{\partial y}\,\mathrm{d}y + \frac{\partial P}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial Q}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y + \frac{\partial Q}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}y \\
&\quad\quad + \left(\frac{\partial R}{\partial x}\,\mathrm{d}x + \frac{\partial R}{\partial y}\,\mathrm{d}y + \frac{\partial R}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}z \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z \wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y.
\end{aligned}$$

Similarly, we can differentiate a 2-form $\omega = P\,\mathrm{d}y \wedge \mathrm{d}z + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}y$ to get:

$$\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

The two results above resemble the curl and divergence of (P, Q, R). A differential form ω is *closed* if $\mathrm{d}\omega = 0$, and is *exact* if there exists η such that $\omega = \mathrm{d}\eta$.

Lemma 1.2.1 *(Poincaré)*: For any differential form ω on an open, contractible set $U \subseteq \mathbb{R}^n$, if ω is closed, then it is also exact.

It is true that for any set $U \subseteq \mathbb{R}^n$, regardless of contractibility, that for a differential form ω defined on U, $\mathrm{d}(\mathrm{d}\omega) = 0$. In other words, all exact differential forms are closed. (For a region U, we have $\partial\partial U = \varnothing$. This is one of many reasons for which the boundary operator is denoted by ∂, in analogy to d.)

The implications of this are important: if ω is a 0-form, then $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\omega) = 0$, and if ω is a 1-form, $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) = 0$, where $\mathbf{v}$ is the vector of the coefficients of the basis differential forms of ω (there are no correlations for higher degree forms since in 3-dimensional space, the highest degree possible for any differential form is 3).

Then, the Fundamental Theorem of Calculus, the Gradient Theorem, Green's, Stokes', and Gauss' Theorems can be generalized into:

Theorem 1.2.11 *(Stokes–Cartan)*: For an oriented smooth n-dimensional compact manifold M with boundary ∂M, for a smooth differential $(n-1)$-form ω over $\overline{M}$,

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

Real analysis is the subject dedicated to rigorously defining concepts such as limits, continuity, integrability, convergence, etc. The most widely used definition of a finite limit of a function is the language of ε–δ, which states:

Definition 1.2.1 *(Epsilon–Delta)*: Let $f : U \to \mathbb{R}$ be a function defined over an open set $U \subseteq \mathbb{R}$ such that a is an accumulation point of U. We say that $\lim_{x \to a} f(x) = L$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for all $x \in U$ with $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon$.

Similarly, we define the *right-handed limit* $\lim_{x \to a^+} f(x) = L$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in U$ with $0 < x - a < \delta$, we have $|f(x) - L| < \varepsilon$.

Likewise, the *left-hand limit* $\lim_{x \to a^-} f(x) = L$ exists if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in U$ with $-\delta < x - a < 0$, we have $|f(x) - L| < \varepsilon$.

We also have the definition of the limit of a sequence:

Definition 1.2.2 *(Epsilon–N)*: Let $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ be a sequence. If $\exists a_\infty \in \mathbb{R}$ such that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|a_n - a_\infty| < \varepsilon$, then $\{a_n\}$ *converges* to a_∞.

Theorem 1.2.12 *(Cauchy Criterion)*: Let $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ be a sequence. Then $\{a_n\}$ is convergent iff $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n, m > N$, $|a_n - a_m| < \varepsilon$.

Proof: Assume $\{a_n\}$ is convergent. Then $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n, m > N$, $|a_n - a_\infty| < \frac{\varepsilon}{2}$ and $|a_m - a_\infty| < \frac{\varepsilon}{2}$ for some $a_\infty \in \mathbb{R}$. It follows that

$$|a_n - a_m| \le |a_n - a_\infty| + |a_m - a_\infty| = \varepsilon.$$

Conversely, $\{a_n\}$ is bounded (fixing N, $\forall n > N$, $\left|a_n - a_{N+1}\right| < \varepsilon$). By the Bolzano–Weierstrass Theorem (Theorem 1.1.2), $\{a_n\}_{n \in \mathbb{N}}$ has a subsequence $\left\{a_{n_k}\right\}_{k \in \mathbb{N}}$ that converges to a_∞. Therefore, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ and $\exists M \in \mathbb{N}$ such that $\forall k > M, n_k > N$, and $\forall n > N$, $\left|a_n - a_{n_k}\right| < \frac{\varepsilon}{2}$ and $\left|a_{n_k} - a_\infty\right| < \frac{\varepsilon}{2}$. Then

$$|a_n - a_\infty| \le \left|a_n - a_{n_k}\right| + \left|a_{n_k} - a_\infty\right| < \varepsilon.$$

Hence, $\{a_n\}$ converges to a_∞. □

Definition 1.2.3 *(Limit Superior)*: For a number sequence $\{a_n\} \subset \mathbb{R}$, if $\exists a \in \mathbb{R}$ such that:

1 $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n > N$, $a_n < a + \varepsilon$,

2 $\forall \varepsilon > 0$, $\forall N \in \mathbb{N}$, $\exists n > N$ such that $a_n > a - \varepsilon$,

then the *superior limit* of $\{a_n\}$ is a, denoted by $\limsup_{n\to\infty} a_n = a$.

Definition 1.2.4 *(Limit Inferior)*: For a number sequence $\{a_n\} \subset \mathbb{R}$, if $\exists a \in \mathbb{R}$ such that:
1 $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $a_n > a - \varepsilon$,
2 $\forall \varepsilon > 0$, $\forall N \in \mathbb{N}$, $\exists n > N$ such that $a_n < a + \varepsilon$,

then the *inferior limit* of $\{a_n\}$ is a, denoted by $\liminf_{n\to\infty} a_n = a$.

Lemma 1.2.2: A number sequence $\{a_n\}$ is convergent iff $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$.

Proof: We first prove that $a = \lim_{n\to\infty} a_n$ implies $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a$. By Definition 1.2.2, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$,

$$|a_n - a| < \varepsilon \Longleftrightarrow a - \varepsilon < a_n < a + \varepsilon.$$

Then from Definition 1.2.3 and Definition 1.2.4, we have that $\limsup_{n\to\infty} a_n \geq a$ and $\liminf_{n\to\infty} a_n \leq a$. By the second conditions, we get $\limsup_{n\to\infty} a_n \leq a$ and $\liminf_{n\to\infty} a_n \geq a$. Therefore,

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n.$$

For the converse, assume $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. Since $\exists N_1 \in \mathbb{N}$ such that $\forall n > N_1$, $a_n < a + \varepsilon$. $\exists N_2 \in \mathbb{N}$ such that $\forall n > N_2$, $a_n > a - \varepsilon$. Then $\forall n > \max\{N_1, N_2\}$, $|a_n - a| < \varepsilon$, as expected. □

Definition 1.2.5 *(Continuity)*: A function $f : U \to \mathbb{R}$, defined on an open set $U \subseteq \mathbb{R}$ containing a point $a \in U$, is said to be continuous at a iff

$$\lim_{x\to a} f(x) = f(a).$$

It is important to note that in the case of multiple *explicit* variables, a distinction is made between (separate) continuity (where there are two δ's on which variable varies, and does not guarantee a single δ for when both variables vary simultaneously) and *joint* continuity (where a single δ controls both variables at once). To illustrate this, let (x_0, y_0) be fixed. The former is commonly written as

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall |x - x_0| < \delta, |f(x, y_0) - f(x_0, y_0)| < \varepsilon$$

in conjunction with

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall |y - y_0| < \delta, |f(x_0, y) - f(x_0, y_0)| < \varepsilon,$$

whereas the latter is expressed as

$\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall |(x - x_0, y - y_0)| < \delta, |f(x, y) - f(x_0, y_0)| < \varepsilon.$

Theorem 1.2.13: Any continuous function on a compact set K is bounded on K.

Proof: Suppose for the sake of contradiction that $f : U \to \mathbb{R}$ is continuous and unbounded on compact K. Then for each $n \in \mathbb{N}$, there exists $x_n \in K$ such that $|f(x_n)| > n$. The sequence $\{x_n\}$ lies in K, which is compact, so by the Bolzano–Weierstrass Theorem (Theorem 1.1.2), $\{x_n\}$ has an accumulation point in K. In other words, there exists a convergent subsequence $\left\{x_{n_k}\right\}$ with $\lim_{k \to \infty} x_{n_k} \in K$.

Since f is continuous, $\lim_{k \to \infty} f\left(x_{n_k}\right) = f\left(\lim_{k \to \infty} x_{n_k}\right)$, which is well-defined because $\lim_{k \to \infty} x_{n_k} \in K$. However, this contradicts $\left|f\left(x_{n_k}\right)\right| > n_k \to \infty$, hence f must be bounded on K. □

Theorem 1.2.14 *(Extreme Value)*: A continuous function $f(x)$ defined on a compact set K attains its infimum and supremum in K.

Proof: Assume that f never attains its supremum M. Then, $f(x) < M$. Define the auxiliary function $\psi(x) = \frac{1}{M - f(x)}$, which is strictly positive and continuous as the denominator never reaches 0. By Theorem 1.2.13, $\psi(x)$ is bounded with some value of $\mu > 0$ satisfying $\psi(x) \leq \mu$. $f(x)$ also has the representation $M - \frac{1}{\psi(x)}$, and therefore,

$$f(x) \leq M - \frac{1}{\mu},$$

which means that M is not the supremum. Similarly, assume that f never attains its infimum m. Then $f(x) > m$. Let $\psi(x) = \frac{1}{f(x) - m}$, which is strictly positive and continuous as the denominator never reaches 0. By Theorem 1.2.13, $\psi(x)$ is bounded with some value of $\mu > 0$ satisfying $\psi(x) \leq \mu$. $f(x)$ also has the representation $m + \frac{1}{\psi(x)}$, and therefore,

$$f(x) \geq m + \frac{1}{\mu},$$

which means that m is not the infimum. □

Definition 1.2.6 *(Uniform Continuity)*: A function $f : U \to \mathbb{R}$, defined on a set $U \subseteq \mathbb{R}$, is uniformly continuous iff $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x, y \in U$ where $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$.

Example 1.2.1: The function $f(x) = \frac{1}{x}$ is not uniformly continuous over $(0, 1)$.

Proof: If $\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists x, y \in (0,1)$ satisfying both $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$, then f is not uniformly continuous over $(0,1)$.

Let $\varepsilon = 1$ and

$$x = \frac{1}{n}, \quad y = \frac{1}{n+1}.$$

Then $\forall \delta > 0$, $\exists n > 1$ where $|x - y| < \delta$, since $\lim_{n \to \infty} |x - y| = 0$. Additionally, $|f(x) - f(y)| = 1 \geq \varepsilon$. This satisfies the negation, and thus, $f(x) = \frac{1}{x}$ is not uniformly continuous over $(0,1)$. ☐

Theorem 1.2.15 *(Heine–Cantor)*: A continuous function on a compact set K is uniformly continuous on K.

Proof: Fix $x \in K$. Since f is continuous at x, for every $\varepsilon > 0$ there exists $\delta_x > 0$ such that for all $\zeta \in D(x, \delta_x) \cap K$,

$$|f(\zeta) - f(x)| < \frac{\varepsilon}{2}. \tag{1.2.1}$$

The collection of open balls $\left\{D\left(x, \frac{\delta_x}{2}\right)\right\}_{x \in K}$ forms an open cover of the compact set K. By Heine–Borel (Theorem 1.1.3), there is a finite subcover

$$\left\{D\left(x_k, \frac{\delta_{x_k}}{2}\right)\right\}_{k=1}^{n}.$$

Set

$$\delta = \min_{1 \leq k \leq n} \frac{\delta_{x_k}}{2}.$$

Now let $x, y \in K$ satisfy $|x - y| < \delta$. Then there exists an index $j \in \{1, \cdots, n\}$ such that $x \in D\left(x_j, \frac{\delta_{x_j}}{2}\right)$. Consequently,

$$|x_j - y| \leq |x_j - x| + |x - y| < \frac{\delta_{x_j}}{2} + \delta \leq \delta_{x_j}.$$

Applying (1.2.1) to the points x and y through x_j, we obtain

$$|f(x_j) - f(x)| < \frac{\varepsilon}{2}, \quad |f(x_j) - f(y)| < \frac{\varepsilon}{2}.$$

Therefore,

$$|f(x) - f(y)| \leq |f(x) - f(x_j)| + |f(x_j) - f(y)| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the uniform continuity of f on K follows. ☐

Definition 1.2.7: A function f is Lipschitz continuous over U if $\exists M \in \mathbb{R}_{\geq 0}$ such that $\forall x, y \in U$, $|f(x) - f(y)| \leq M|x - y|$. The smallest possible M satisfying the above condition is known as the Lipschitz constant.

Lipschitz continuity is an important concept in real analysis and the theory of differential equations. It is a strong form of uniform continuity.

Proposition 1.2.1: All Lipschitz continuous functions on U are uniformly continuous on U.

Proof: Let $M > 0$ be the Lipschitz constant. Then $\forall \varepsilon > 0$, let $\delta = \frac{\varepsilon}{M}$. It then follows that $\forall x, y \in U$ such that $|x - y| < \delta$, $|f(x) - f(y)| \leq M|x - y| < \varepsilon$. □

Proposition 1.2.2: A C^1 function on a compact set K is Lipschitz continuous on K.

Proof: Let $f : K \to \mathbb{R}$ be C^1. By Theorem 1.2.13, since K is compact and f' is continuous, $\exists M > 0$ such that $\forall x \in K$, $|f'(x)| \leq M$.

By the Mean Value Theorem, $\forall x, y \in K$, $\exists c$ between x and y such that $f(x) - f(y) = f'(c)(x - y)$. Then, $|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|$, which means f is Lipschitz continuous with Lipschitz constant less than or equal to M. □

2 Complex Prerequisites

2.1 The Extended Complex Plane and its Spherical Representation

All complex numbers form a field that extends the real number field. A complex number $\alpha + \mathrm{i}\beta$ can be visualized on a rectangular plane as the point (α, β), with two axes: the real axis and the imaginary axis. It is well known that any complex number also has the polar form $r\mathrm{e}^{\mathrm{i}\theta} = r(\cos\theta + \mathrm{i}\sin\theta)$.

The point at infinity, ∞, extends $\mathbb{C}$ to

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

The following arithmetic operations are defined: for all $a \in \mathbb{C}$,

$$a + \infty = \infty + a = \infty,$$

and for all $b \in \mathbb{C} \setminus \{0\}$,

$$b \cdot \infty = \infty \cdot b = \infty, \quad \frac{a}{\infty} = 0.$$

Let

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

There exists a *stereographic projection* of S^2 onto $\hat{\mathbb{C}}$. For every point other than $(0, 0, 1)$, there is a corresponding complex number

$$z = \frac{x_1 + \mathrm{i}x_2}{1 - x_3}. \tag{2.1.1}$$

This correspondence between $\mathbb{C}$ and $S^2 \setminus \{(0, 0, 1)\}$ is injective. In fact, the inverse can be solved for:

$$|z|^2 = \frac{1 - x_3^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3},$$

which results in

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1},$$

and consequently,

$$x_1 = \mathfrak{Re}(z)(1 - x_3) = \frac{z + \overline{z}}{|z|^2 + 1},$$

$$x_2 = \mathfrak{Im}(z)(1 - x_3) = \frac{z - \overline{z}}{\mathrm{i}|z|^2 + \mathrm{i}}.$$

By letting ∞ correspond to $(0, 0, 1)$, the bijection is complete. The sphere S^2 is also called the *Riemann sphere*. The region given by the disk $|z| < 1$ corresponds to $x_3 < 0$, and the region $|z| > 1$ corresponds to $x_3 > 0$.

We will now give a geometric visualization of this projection. Let $z = x + \mathrm{i}y$. Then we obtain

$$x = \frac{x_1}{1 - x_3} \quad \text{and} \quad y = \frac{x_2}{1 - x_3}.$$

Therefore,

$$x : y : 1 = x_1 : x_2 : 1 - x_3.$$

It follows that the points $(0, 0, 0)$, $(x, y, 1)$, and $(x_1, x_2, 1 - x_3)$ are collinear in $\mathbb{R}^3$. Under the linear map

$$\mathbf{v} \mapsto \mathbf{v} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + (0, 0, 1),$$

we get that $(0,0,1)$, $(x,y,0)$, and (x_1,x_2,x_3) are collinear. In other words, this correspondence is a central projection with center $(0,0,1)$, projecting the points from $S^2 \setminus (0,0,1)$ onto $\mathbb{C}$. Let this center correspond to ∞. In this representation, $\infty \in \hat{\mathbb{C}}$ is no longer considered to be "special".

It is worth noting that in several geometric contexts, an alternative paradigm exists where we let S be the sphere centered at $\left(0,0,\frac{1}{2}\right)$ of diameter 1, and project points from the north pole $(0,0,1)$ onto the horizontal plane of tangency. Later in Nevanlinna theory, specifically in @ sec:nevanlinnatheory, we will observe that in some sense this is the more natural object to study. The corresponding equations are then

$$x_1 = \frac{\mathfrak{Re}(z)}{|z|^2+1}, \quad x_2 = \frac{\mathfrak{Im}(z)}{|z|^2+1}, \quad x_3 = \frac{|z|^2}{|z|^2+1}.$$

The forward projection remains unchanged. Lastly, we define the upper half-plane; for the following sections, let

$$\mathbb{H}^+ = \{z \in \mathbb{C} : \mathfrak{Im}(z) > 0\}$$

2.2 Complex Differentiation

For $U \subseteq \mathbb{C}$ and a complex function $f : U \to \mathbb{C}$, $f(z)$ is *complex differentiable* at $z \in U$ if the following limit exists, regardless of the direction Δz approaches 0 from:

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}.$$

We can consider $f(z)$ to be a bivariate function $f(x,y)$ for $z = x + \mathrm{i}y$. Two main cases we are concerned with are when Δz approaches 0 from the real and imaginary axes:

$$\lim_{\substack{\Delta z \to 0 \\ \Delta z \in \mathbb{R}}} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\substack{\Delta z \to 0 \\ \Delta z \in \mathbb{R}}} \frac{f(z+\mathrm{i}\Delta z) - f(z)}{\mathrm{i}\Delta z}.$$

Expressing $f(z)$ as $f(x,y) = u(x,y) + \mathrm{i}v(x,y)$,

$$\lim_{\substack{\Delta z \to 0 \\ \Delta z \in \mathbb{R}}} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\substack{\Delta z \to 0 \\ \Delta z \in \mathbb{R}}} \frac{f(x+\Delta z, y) - f(x,y)}{\Delta z} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x},$$

and

$$\lim_{\substack{\Delta z \to 0 \\ \Delta z \in \mathbb{R}}} \frac{f(z+\mathrm{i}\Delta z) - f(z)}{\mathrm{i}\Delta z} = -\mathrm{i} \lim_{\substack{\Delta z \to 0 \\ \Delta z \in \mathbb{R}}} \frac{f(x, y+\Delta z) - f(x,y)}{\Delta z} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y}.$$

By comparing the real and imaginary parts, we obtain necessary conditions for complex differentiability:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{2.2.1}$$

By multiplying the second equation by i and adding it to the first, we obtain the equivalent form

$$\frac{\partial f}{\partial x} = -\mathrm{i}\frac{\partial f}{\partial y}. \tag{2.2.2}$$

The identities (2.2.1) and (2.2.2) are known as the *Cauchy–Riemann equations*.

Although this condition is necessary, it is not sufficient. Consider the function

$$f(z) = \sqrt{|\mathfrak{Re}(z)\,\mathfrak{Im}(z)|}.$$

Let $z = x + \mathrm{i}y$, $x = \alpha t$, and $y = \beta t$. Then

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{t \to 0} \frac{\sqrt{|\alpha\beta t^2|}}{\alpha t + \mathrm{i}\beta t} = \frac{\sqrt{|\alpha\beta|}}{\alpha + \mathrm{i}\beta}.$$

The derivative along $\alpha = 1$, $\beta = 0$, or the real axis, vanishes. Along $\alpha = 0$, $\beta = 1$, or the imaginary axis, it also vanishes. However, the limit is different for any other pair of α and β, and hence for other directions of approach.

Definition 2.2.1 *(Holomorphy)*: A function $f : U \to \mathbb{C}$ is said to be *holomorphic* at $z_0 \in U$ if it is complex differentiable on a neighborhood of z_0. If $f(z)$ is holomorphic for every point in an open connected set U, then it is said to be holomorphic over U. A function is holomorphic over a compact set K if it is holomorphic on a neighborhood of K.

Weierstrass provided the following classification:

Definition 2.2.2: A function is *entire* if it is holomorphic over $\mathbb{C}$.

For the purpose of the following contents, a *region* or *domain* will denote a nonempty, open, connected subset of the complex plane.

Theorem 2.2.1: Let $U \subseteq \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$ be a function. Then f is holomorphic on U iff $f \in C^1(U)$ and satisfies the Cauchy–Riemann equations.

Proof: The first part is to prove that any holomorphic function on U has continuous first-order partial derivatives in U. This requires an argument that will be covered later, specifically in Section 3.2, which states that the complex derivative of any holomorphic function is also holomorphic over the region.

For the second part, let $f(z) = f(x, y) = u(x, y) + \mathrm{i}v(x, y)$. Assume that $u, v \in C^1(\{z_0\})$ and satisfy the Cauchy–Riemann equations at $z_0 = x_0 + \mathrm{i}y_0$. Let

$$\alpha = \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \beta = \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0).$$

Then because $u, v \in C^1(U)$ have continuous partial derivatives, $\forall x + \mathrm{i}y \in U$:

$$u(x, y) - u(x_0, y_0) = \alpha(x - x_0) - \beta(y - y_0) + o(|\Delta z|),$$

$$v(x, y) - v(x_0, y_0) = \beta(x - x_0) + \alpha(y - y_0) + o(|\Delta z|),$$

where $|\Delta z| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ and $o(|\Delta z|)$ denotes a value with higher infinitesimal order to $|\Delta z|$, or that $\lim_{\Delta z \to 0} \frac{o(|\Delta z|)}{|\Delta z|} = 0$. Then letting $\Delta z = x - x_0 + \mathrm{i}(y - y_0)$,

$$f(z) - f(z_0) = \alpha \Delta z + \mathrm{i}\beta \Delta z + o(|\Delta z|) + o(|\Delta z|),$$

$$\frac{f(z) - f(z_0)}{z - z_0} = \alpha + \mathrm{i}\beta + \frac{o(|\Delta z|)}{|\Delta z|} \cdot \frac{|\Delta z|}{\Delta z}.$$

Taking the limit as $\Delta z \to 0$, the high order infinitesimals on the right-hand side vanish, and

$$\lim_{\Delta z \to 0} \frac{f(z) - f(z_0)}{z - z_0} = \alpha + \mathrm{i}\beta. \qquad \square$$

We will prove later in Section 3.2 that the complex derivative of a holomorphic function $f(z) = u(z) + \mathrm{i}v(z)$ is holomorphic. Under this assumption, $f(z)$ has continuous second-order partial derivatives, and therefore,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x},$$

and by the Cauchy–Riemann equations,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x},$$

and

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x}.$$

Adding the equations,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

This general type of equation is known as *Laplace's equation*, which is a basic example of an elliptic partial differential equation. Define the operator (the *Laplacian*)

$$\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

A function u satisfying Laplace's equation $\nabla^2 u = 0$ is a *harmonic function.* Thus, the real and complex parts of a holomorphic function are harmonic functions.

Letting $x = r\cos\theta$, $y = r\sin\theta$, the Laplacian is equal to:

$$\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\right) + \frac{\partial}{\partial y}\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{x}{r}\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial \theta}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\frac{\partial}{\partial r} + \frac{x}{r^2}\frac{\partial}{\partial \theta}\right) \\
&= \left(\frac{x}{r}\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial \theta}\right)\left(\frac{x}{r}\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial \theta}\right) + \left(\frac{y}{r}\frac{\partial}{\partial r} + \frac{x}{r^2}\frac{\partial}{\partial \theta}\right)\left(\frac{y}{r}\frac{\partial}{\partial r} + \frac{x}{r^2}\frac{\partial}{\partial \theta}\right) \\
&= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right) \\
&\quad + \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right) \\
&= \cos^2\theta\frac{\partial^2}{\partial r^2} - \cos\theta\sin\theta\left(-\left(\frac{1}{r^2}\right)\frac{\partial}{\partial \theta} + \left(\frac{1}{r}\right)\frac{\partial^2}{\partial \theta \partial r}\right) \qquad (2.2.3) \\
&\quad - \frac{\sin\theta}{r}\left(-\sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{\partial^2}{\partial \theta \partial r}\right) + \frac{\sin\theta}{r^2}\left(\cos\theta\frac{\partial}{\partial \theta} + \sin\theta\frac{\partial^2}{\partial \theta^2}\right) \\
&\quad + \sin^2\theta\frac{\partial^2}{\partial r^2} + \sin\theta\cos\theta\left(-\left(\frac{1}{r^2}\right)\frac{\partial}{\partial \theta} + \left(\frac{1}{r}\right)\frac{\partial^2}{\partial \theta \partial r}\right) \\
&\quad + \frac{\cos\theta}{r}\left(\cos\theta\frac{\partial}{\partial r} + \sin\theta\frac{\partial^2}{\partial \theta \partial r}\right) + \frac{\cos\theta}{r^2}\left(-\sin\theta\frac{\partial}{\partial \theta} + \cos\theta\frac{\partial^2}{\partial \theta^2}\right) \\
&= \frac{\partial^2}{\partial r^2} + \left(\frac{1}{r^2}\right)\frac{\partial^2}{\partial \theta^2} + \left(\frac{1}{r}\right)\frac{\partial}{\partial r}.
\end{aligned}$$

Proposition 2.2.1: Let $U \subseteq \mathbb{C}$ be open and connected and let $f : U \to \mathbb{R}$ be holomorphic. It follows that f is constant over U.

Proof: Since $f(x,y) = u(x,y) + \mathrm{i}v(x,y)$ is real-valued, $v(x,y) \equiv 0$ on U. Then by the Cauchy–Riemann equations on U, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$. Similarly, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$. Therefore, $f(z) = u(z)$ is constant. □

2.2.1 Wirtinger Derivatives

We have previously introduced the concept of expressing a complex function as a function of x and y. It can also be expressed in terms of z and $\overline{z}$, where $z = x + \mathrm{i}y$ and $\overline{z} = x - \mathrm{i}y$. Then $|z|^2 = z\overline{z}$, $x = \frac{z+\overline{z}}{2}$, and $y = \frac{z-\overline{z}}{2\mathrm{i}}$. By the rules of the derivative, it is only natural that we define

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right) \tag{2.2.4}$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right). \tag{2.2.5}$$

If (2.2.4) is set equal to 0, then it is the equivalent form of the homogeneous Cauchy–Riemann Equations. Then for a holomorphic function $f(z)$, the Wirtinger derivative $\frac{\partial f}{\partial z} = \frac{\mathrm{d}f}{\mathrm{d}z}$.

In terms of u and v, the two derivatives of a function $f(z)$ are equal to:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right),$$

and

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} + \mathrm{i}\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right).$$

If f is holomorphic,

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y}. \tag{2.2.6}$$

On the contrary, by the rules of the derivative,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}$$

and

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial y} = \mathrm{i}\frac{\partial}{\partial z} - \mathrm{i}\frac{\partial}{\partial \overline{z}}.$$

The Laplacian is equal to

$$
\begin{aligned}
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} &= \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\right)^2 + \left(\mathrm{i}\frac{\partial}{\partial z} - \mathrm{i}\frac{\partial}{\partial \overline{z}}\right)^2 \\
&= \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \overline{z}^2} + 2\frac{\partial^2}{\partial z \partial \overline{z}} \\
&\quad - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \overline{z}^2} + 2\frac{\partial^2}{\partial z \partial \overline{z}} \\
&= 4\frac{\partial^2}{\partial z \partial \overline{z}}.
\end{aligned}
\tag{2.2.7}
$$

Under this definition, we can derive the chain rule:

Theorem 2.2.1.1 *(Chain Rule)*: Let $\Omega \subseteq \mathbb{C}$ be a region such that $g \in C^1(\Omega)$ and $f \in C^1(g(\Omega))$. Writing $w = g(z)$, it follows that

$$
\begin{aligned}
\frac{\partial}{\partial z}(f \circ g) &= \left(\frac{\partial f}{\partial w} \circ g\right)\frac{\partial g}{\partial z} + \left(\frac{\partial f}{\partial \overline{w}} \circ g\right)\frac{\partial \overline{g}}{\partial z} \\
\frac{\partial}{\partial \overline{z}}(f \circ g) &= \left(\frac{\partial f}{\partial w} \circ g\right)\frac{\partial g}{\partial \overline{z}} + \left(\frac{\partial f}{\partial \overline{w}} \circ g\right)\frac{\partial \overline{g}}{\partial \overline{z}}.
\end{aligned}
$$

Proof: Write $z = x + \mathrm{i}y$. Let

$$g(z) = \xi(x, y) + \mathrm{i}\eta(x, y), \quad w = \xi + \mathrm{i}\eta$$

so that $w = g(z)$ with $\xi = \xi(x, y), \eta = \eta(x, y)$. Let f be regarded as a C^1 function of the real variables ξ, η; equivalently we may view f as $f(w, \overline{w})$ where $\overline{w} = \xi - \mathrm{i}\eta$. The composition is $h(z) = f \circ g(z) = f(\xi(x, y), \eta(x, y))$.

Using the real chain rule (provided by the continuous differentiability), we have

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}, \quad \frac{\partial h}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}.$$

Hence,

$$\frac{\partial h}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial \xi}\left(\frac{\partial \xi}{\partial x} - \mathrm{i}\frac{\partial \xi}{\partial y}\right) + \frac{\partial f}{\partial \eta}\left(\frac{\partial \eta}{\partial x} - \mathrm{i}\frac{\partial \eta}{\partial y}\right)\right].$$

Now recall

$$\frac{\partial f}{\partial w} = \frac{1}{2}\left(\frac{\partial f}{\partial \xi} - \mathrm{i}\frac{\partial f}{\partial \eta}\right), \quad \frac{\partial f}{\partial \overline{w}} = \frac{1}{2}\left(\frac{\partial f}{\partial \xi} + \mathrm{i}\frac{\partial f}{\partial \eta}\right).$$

Thus,

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial w} + \frac{\partial f}{\partial \overline{w}}, \quad \frac{\partial f}{\partial \eta} = \mathrm{i}\left(\frac{\partial f}{\partial w} - \frac{\partial f}{\partial \overline{w}}\right).$$

Then by substitution,

$$\begin{aligned}\frac{\partial h}{\partial z} &= \frac{1}{2}\left[\left(\frac{\partial f}{\partial w} + \frac{\partial f}{\partial \overline{w}}\right)\left(\frac{\partial \xi}{\partial x} - \mathrm{i}\frac{\partial \xi}{\partial y}\right)\right.\\ &\qquad \left. + \mathrm{i}\left(\frac{\partial f}{\partial w} - \frac{\partial f}{\partial \overline{w}}\right)\left(\frac{\partial \eta}{\partial x} - \mathrm{i}\frac{\partial \eta}{\partial y}\right)\right]\\ &= \frac{\partial f}{\partial w}\frac{1}{2}\left[\left(\frac{\partial \xi}{\partial x} - \mathrm{i}\frac{\partial \xi}{\partial y}\right) + \mathrm{i}\left(\frac{\partial \eta}{\partial x} - \mathrm{i}\frac{\partial \eta}{\partial y}\right)\right]\\ &\qquad + \frac{\partial f}{\partial \overline{w}}\frac{1}{2}\left[\left(\frac{\partial \xi}{\partial x} - \mathrm{i}\frac{\partial \xi}{\partial y}\right) - \mathrm{i}\left(\frac{\partial \eta}{\partial x} - \mathrm{i}\frac{\partial \eta}{\partial y}\right)\right].\end{aligned}$$

The terms in brackets equal $\frac{\partial g}{\partial z}$ and $\frac{\partial \overline{g}}{\partial z}$. Thus,

$$\frac{\partial h}{\partial z} = \left(\frac{\partial f}{\partial w} \circ g\right)\frac{\partial g}{\partial z} + \left(\frac{\partial f}{\partial \overline{w}} \circ g\right)\frac{\partial \overline{g}}{\partial z}.$$

A similar calculation using (2.2.5) gives

$$\frac{\partial}{\partial \overline{z}}(f \circ g) = \left(\frac{\partial f}{\partial w} \circ g\right)\frac{\partial g}{\partial \overline{z}} + \left(\frac{\partial f}{\partial \overline{w}} \circ g\right)\frac{\partial \overline{g}}{\partial \overline{z}}.$$

These are exactly the proclaimed identities. □

Last we have taking derivatives of conjugates:

Theorem 2.2.1.2: Let $f \in C^1(\Omega)$ where $\Omega \subseteq \mathbb{C}$ is a region. Then

$$\frac{\partial \overline{f}}{\partial z} = \overline{\frac{\partial f}{\partial \overline{z}}}, \quad \frac{\partial \overline{f}}{\partial \overline{z}} = \overline{\frac{\partial f}{\partial z}}.$$

Proof: Write $z = x + \mathrm{i}y$ and $f(z) = u(x,y) + \mathrm{i}v(x,y)$ with $u, v \in C^1(\Omega)$. Then $\overline{f(z)} = u(x,y) - \mathrm{i}v(x,y)$. We compute

$$\frac{\partial \overline{f}}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)(u - \mathrm{i}v) = \frac{1}{2}\big(u_x' - v_y' - \mathrm{i}\big(v_x' + u_y'\big)\big).$$

On the other hand,

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)(u + \mathrm{i}v) = \frac{1}{2}\big(u_x' - v_y' + \mathrm{i}\big(v_x' + u_y'\big)\big).$$

Taking complex conjugates yields

$$\overline{\frac{\partial f}{\partial \bar{z}}} = \frac{1}{2}\big(u'_x - v'_y - \mathrm{i}\big(v'_x + u'_y\big)\big) = \frac{\partial \bar{f}}{\partial z}.$$

Similarly,

$$\frac{\partial \bar{f}}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)(u - \mathrm{i}v) = \frac{1}{2}\big(u'_x + v'_y + \mathrm{i}\big(u'_y - v'_x\big)\big),$$

while

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)(u + \mathrm{i}v) = \frac{1}{2}\big(u'_x + v'_y + \mathrm{i}\big(v'_x - u'_y\big)\big).$$

Taking complex conjugates gives

$$\overline{\frac{\partial f}{\partial z}} = \frac{1}{2}\big(u'_x + v'_y + \mathrm{i}\big(u'_y - v'_x\big)\big) = \frac{\partial \bar{f}}{\partial \bar{z}}. \qquad \square$$

2.3 Complex Power Series

Power series in real analysis can be generalized into complex series. Particularly, concepts such as uniform convergence are analogous to those in real analysis:

Definition 2.3.1 *(Uniform Convergence)*: For a set $U \subseteq \mathbb{C}$, a function sequence $\{f_n(z)\}$ *uniformly converges* to a function $f(z)$ on U iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $\forall z \in U$, $|f_n(z) - f(z)| < \varepsilon$.

Remark: The definition above is equivalent to the following definition.

For a set $U \subseteq \mathbb{C}$, a function sequence $\{f_n(z)\}$ uniformly converges to $f(z)$ iff

$$\lim_{n \to \infty} \sup_{z \in U} |f_n(z) - f(z)| = 0.$$

Informally, we will use the notation $f_n(z) \rightrightarrows f(z)$ to represent uniform convergence.

Theorem 2.3.1 *(Cauchy Criterion)*: For a set $U \subseteq \mathbb{C}$, a function sequence $\{f_n(z)\}$ uniformly converges to a function $f(z)$ iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m > N$, $\forall z \in U$, $|f_n(z) - f_m(z)| < \varepsilon$.

Proof: If $f_n(z)$ uniformly converges to $f(z)$, then for $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m > N$ and $\forall z \in U$,

$$|f_m(z) - f(z)| < \frac{\varepsilon}{2}, \quad |f_n(z) - f(z)| < \frac{\varepsilon}{2}.$$

Then,

$$|f_m(z) - f_n(z)| \leq |f_n(z) - f(z)| + |f_m(z) - f(z)| < \varepsilon.$$

For the converse, refer to the analogous proof in Theorem 1.2.12. □

Function series are defined to be a sequence formed by the partial sums of function sequences. There are many ways to verify the uniform convergence of a function series. Perhaps the most widely known is the Weierstrass M -Test.

Theorem 2.3.2 *(Weierstrass M-Test)*: Let $U \subseteq \mathbb{C}$ be a region and $\{f_n\}$ be a function sequence on U.

If $\exists \{M_n\} \subset \mathbb{R}_{\geq 0}$ such that $\forall n \in \mathbb{N}$, $\forall z \in U$, $|f_n(z)| \leq M_n$ and the series $\sum_{n=1}^{\infty} M_n$ converges, then the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly and absolutely on U.

Proof: By the convergence of $\sum_{n=1}^{\infty} M_n$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m \geq n > N$,

$$\left|M_m + M_{m-1} + \dots + M_{n+1}\right| < \varepsilon.$$

Since M_j bounds $f_j(z)$, it follows that

$$\left|f_m(z) + f_{m-1}(z) + \dots + f_{n+1}(z)\right| \leq \left|M_m + M_{m-1} + \dots + M_{n+1}\right| < \varepsilon,$$

and the result follows from Theorem 2.3.1. □

The concept of uniform convergence is generalized to improper integrals with parameters, and the same theorems from series have a corresponding counterpart.

In both complex and real analysis, the concept of *power series*, a unique type of function series, is of trivial importance. Similar to real power series, complex series have the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $\{a_n\}$ are constants.

Let $D(a, r) = B^1(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ denote the *open disk* centered at a with radius r. For simplicity, from now on we will have $\mathbb{D}$ denote the unit open disk, or $D(0, 1)$. We will now observe the convergence of power series.

Theorem 2.3.3 *(Abel's Theorem)*: For a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, there exists a constant $R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, known as the *radius of convergence* such that:

1. f absolutely converges on $D(0, R)$, and $\forall 0 \le \rho < R$, uniformly converges on $\overline{D(0, \rho)}$.
2. $f(z)$ diverges when $|z| > R$.
3. f is holomorphic over $D(0, R)$ and $f'(z)$ can be obtained by termwise differentiation, or $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$, which also has a convergence radius of R.

The disk $|z| < R$ is known as the *disk of convergence*, a direct generalization of the *interval of convergence* for real series. There are many ways to determine the radius of convergence:

Theorem 2.3.4 *(Cauchy–Hadamard)*: The radius of convergence of the power series in the form $\sum_{n=0}^{\infty} a_n z^n$ can be determined by

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}. \tag{2.3.1}$$

Of course, a convergence radius of 0 implies that the series is divergent everywhere except for possibly at 0, and a convergence radius of ∞ means that the series absolutely converges everywhere.

Proof: We will prove that the value in (2.3.1) satisfies the criteria in Theorem 2.3.3.

Assume $|z| < R$. Then $\forall \rho \in (|z|, R)$, and consequently, $\frac{1}{\rho} > \frac{1}{R}$. By Definition 1.2.3 and (2.3.1), $\exists N \in \mathbb{N}$ such that $\forall n > N$, $\sqrt[n]{|a_n|} < \frac{1}{\rho}$ and $|a_n| < \frac{1}{\rho^n}$. It follows that $|a_n z^n| < \frac{|z|^n}{\rho^n} < 1$ for all $n > N$. Then $\sum_{n=0}^{\infty} |a_n z^n|$ converges.

Let $\rho' \in (\rho, R)$. Similarly, $\exists N' \in \mathbb{N}$ such that $\forall n > N'$, $\sqrt[n]{|a_n|} < \frac{1}{\rho'}$, and $|a_n| < \frac{1}{\rho'^n}$. Then $|a_n z^n| < |a_n \rho^n| < \frac{\rho^n}{\rho'^n}$. By the Weierstrass M-Test (Theorem 2.3.2), $\sum_{n=0}^{\infty} |a_n z^n|$ is uniformly bounded for $n > N'$ by the convergent series $\sum_{n=0}^{\infty} a_n \rho^n$, and thus uniformly converges on $|z| < \rho$. This proves part 1.

Assume that $|z| > R$. For all $\rho \in (R, |z|)$, $\frac{1}{\rho} < \frac{1}{R}$. By Definition 1.2.3, $\forall N \in \mathbb{N}$, $\exists n_N > N$ such that $\sqrt[n_N]{|a_{n_N}|} > \frac{1}{\rho}$. It follows that $\left|a_{n_N} z^{n_N}\right| > \frac{|z^{n_N}|}{\rho^{n_N}} > 1$. Since $\forall N \in \mathbb{N}$,

$$\left| \sum_{k=0}^{n_N} a_k z^k - \sum_{k=0}^{n_N - 1} a_k z^k \right| > 1,$$

by the Cauchy Criterion (Theorem 1.2.12), $\sum_{n=0}^{\infty} a_n z^n$ is divergent. Thus, part 2 is satisfied.

To prove part 3, first observe that $\sum_{n=1}^{\infty} na_n z^n$ and $\sum_{n=1}^{\infty} a_n z^n$ have the same convergence radius since $\limsup_{n\to\infty} \sqrt[n]{n} = 1$. For $z \in D(0,R)$, let $f(z) = S_n(z) + R_n(z)$, where

$$S_n(z) = \sum_{k=0}^{n-1} a_k z^k, \quad R_n(z) = \sum_{k=n}^{\infty} a_k z^k.$$

Let

$$f_1(z) = \lim_{n\to\infty} S_n'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Let $\rho < R$ be positive and $|z_0| < \rho$. Then we aim to show that

$$\lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0} - f_1(z)\right) = 0.$$

By analyzing the difference,

$$\begin{aligned}\frac{f(z)-f(z_0)}{z-z_0} - f_1(z) = &\left[\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z)\right] \\ &+ S_n'(z) - f_1(z) + \frac{R_n(z)-R_n(z_0)}{z-z_0}.\end{aligned} \tag{2.3.2}$$

Since $S_n'(z) \to f_1(z)$ as $n\to\infty$, it follows that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|S_n'(z) - f_1(z)| < \frac{\varepsilon}{3}$. Since

$$\frac{R_n(z)-R_n(z_0)}{z-z_0} = \sum_{k=n}^{\infty} a_k\left(z^{k-1} + z^{k-2}z_0 + \cdots + z_0^{k-1}\right)$$

for $z \neq z_0$, with $|z| < \rho < R$,

$$\left|\sum_{k=n}^{\infty} a_k\left(z^{k-1} + \cdots + z_0^{k-1}\right)\right| \le \sum_{k=n}^{\infty} |a_k|\left(\left|z^{k-1}\right| + \cdots + \left|z_0^{k-1}\right|\right) < \sum_{k=n}^{\infty} |a_k| k\rho^{k-1}.$$

Since $\sum_{k=1}^{\infty} k|a_k|\rho^{k-1}$ is absolutely convergent, $\sum_{k=n}^{\infty} |a_k| k\rho^{k-1}$ is the remainder term of a convergent series. Then, $\exists N' \in \mathbb{N}$ such that $\forall n > N'$, $\sum_{k=n}^{\infty} |a_k| k\rho^{k-1} < \frac{\varepsilon}{3}$.

Finally, for a fixed $n > \max(N, N')$, $\exists \delta > 0$ such that $\forall z \in D(z_0,\delta) \setminus \{z_0\}$,

$$\left|\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z)\right| < \frac{\varepsilon}{3}.$$

From (2.3.2), we get

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-f_1(z)\right|<\varepsilon,$$

confirming part 3. ☐

Obviously, a substitution of $z = \zeta - a$ where $a \in \mathbb{C}$ translates the disk of convergence to $D(a, R)$. The subsequent results on uniform convergence hold for complex functions:

Theorem 2.3.5 *(UNIFORM LIMIT)*: Let $\{f_n(z)\}_n$ be a sequence of continuous functions on $U \subseteq \mathbb{C}$ and uniformly convergent to $f(z)$. Then $f(z)$ is continuous on U.

Proof: Fix $\varepsilon > 0$ and $z_0 \in U$. By uniform convergence, $\exists n \in \mathbb{N}$ such that $\forall z \in U$, $|f_n(z) - f(z)| < \frac{\varepsilon}{3}$ and $|f_n(z_0) - f(z_0)| < \frac{\varepsilon}{3}$. By continuity, $\exists \delta > 0$ such that $\forall z \in D(z_0, \delta) \cap U \subseteq U$, $|f_n(z) - f_n(z_0)| < \frac{\varepsilon}{3}$.

By the triangle inequality,

$$|f(z)-f(z_0)| \le |f(z)-f_n(z)| + |f_n(z)-f_n(z_0)| + |f_n(z_0)-f(z_0)| < \varepsilon.$$

for all $z \in D(z_0, \delta) \cap U$. Then the continuity of f is satisfied. ☐

Lastly, the sufficient criteria to pass a limit through an integral:

Theorem 2.3.6: Let γ be a rectifiable curve on which the function sequence $\{f_n\}_{n\in\mathbb{N}}$ is continuous. If $\{f_n(z)\}$ uniformly converges to f, then

$$\lim_{n\to\infty}\int_\gamma f_n(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z.$$

Proof: Since $\{f_n(z)\}$ uniformly converges to $f(z)$ on γ, $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$|f_n(z) - f(z)| < \frac{\varepsilon}{\operatorname{length}(\gamma)}, \quad \forall z \in \gamma.$$

Since each f_n is continuous and γ is rectifiable, each integral $\int_\gamma f_n(z)\,\mathrm{d}z$ is convergent and well-defined.

Then $\forall n > N$,

$$\begin{aligned}\left|\int_\gamma f_n(z)\,\mathrm{d}z-\int_\gamma f(z)\,\mathrm{d}z\right| &= \left|\int_\gamma (f_n(z)-f(z))\,\mathrm{d}z\right| \\ &\le \int_\gamma |f_n(z)-f(z)||\mathrm{d}z| \\ &< \int_\gamma \frac{\varepsilon}{\mathrm{length}(\gamma)}|\mathrm{d}z| = \varepsilon.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\int_\gamma f_n(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z. \qquad \square$$

Remark: For a uniformly convergent series $\sum_{n=1}^{\infty} f_n(z)$, the commutation between the limit and the integral becomes a summation-integral switch:

$$\sum_{n=1}^{\infty}\int_\gamma f_n(z)\,\mathrm{d}z = \int_\gamma \sum_{n=1}^{\infty} f_n(z)\,\mathrm{d}z.$$

2.4 The Conformality of Holomorphic Mappings

Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open and connected subset $U \subseteq \mathbb{C}$, and let $z_0 \in U$ be a point such that $f'(z_0) \neq 0$. Consider a differentiable curve $\gamma \in C^1([0,1])$ with $\gamma(0) = z_0$. The direction of the curve at z_0 is given by the argument of its derivative, namely $\mathrm{Arg}(\gamma'(0))$.

The image of γ under f, defined by $\sigma(t) = f(\gamma(t))$, is also a smooth curve passing through $w_0 = f(z_0)$. By the chain rule, the derivative of σ at $t = 0$ is given by

$$\sigma'(0) = f'(\gamma(0))\gamma'(0) = f'(z_0)\gamma'(0),$$

and hence

$$\mathrm{Arg}(\sigma'(0)) = \mathrm{Arg}(f'(z_0)\gamma'(0)) = \mathrm{Arg}(f'(z_0)) + \mathrm{Arg}(\gamma'(0)).$$

It follows that

$$\mathrm{Arg}(\sigma'(0)) - \mathrm{Arg}(\gamma'(0)) = \mathrm{Arg}(f'(z_0)).$$

This shows that the change in direction between a curve and its image under f is independent of the curve itself, depending only on the value of $f'(z_0)$.

Now consider two smooth curves $\gamma_1, \gamma_2 \in C^1([0,1])$ such that $\gamma_1(0) = \gamma_2(0) = z_0$, with respective images $\sigma_1(t) = f(\gamma_1(t))$ and $\sigma_2(t) = f(\gamma_2(t))$. Then

$$\operatorname{Arg}(\sigma_{1'}(0)) - \operatorname{Arg}(\gamma_{1'}(0)) = \operatorname{Arg}(f'(z_0)) = \operatorname{Arg}(\sigma_{2'}(0)) - \operatorname{Arg}(\gamma_{2'}(0)),$$

and by rearrangement,

$$\operatorname{Arg}(\sigma_{1'}(0)) - \operatorname{Arg}(\sigma_{2'}(0)) = \operatorname{Arg}(\gamma_{1'}(0)) - \operatorname{Arg}(\gamma_{2'}(0)).$$

This equality demonstrates that the angle between two smooth curves at z_0 is preserved under f, provided $f'(z_0) \neq 0$. In other words, holomorphic functions with non-vanishing derivatives preserve angles and orientation locally, a property known as *conformality*.

Furthermore, for any smooth curve $\gamma \in C^1([0,1])$ passing through z_0, the limit

$$\lim_{z \to z_0, z \in \gamma} \frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)|$$

shows that infinitesimal lengths are locally scaled by a factor of $|f'(z_0)|$ under the mapping f.

Therefore, at points where $|f'(z)| \neq 0$, the function f is conformal; it preserves angles but not necessarily lengths.

2.5 Elementary Functions

Functions of one complex variable that are formed by compositions, sums, products, and powers of finitely many functions of the following form are known as *elementary functions*:

1 Power functions including polynomials, rational functions, and their inverses.
2 Trigonometric functions, hyperbolic functions, and their inverses.
3 Exponential functions and their inverses.

Power functions are easily extendable to the complex plane by simply changing the real variable to a complex variable. The other two classes of functions have to be redefined and reinterpreted for the complex plane. It is well known that the exponential function can be expanded as

$$\begin{aligned} \mathrm{e}^x &= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ &= \frac{x^0}{0!\mathrm{i}^0} + \mathrm{i}\frac{x^1}{1!\mathrm{i}^1} - \frac{x^2}{2!\mathrm{i}^2} - \mathrm{i}\frac{x^3}{3!\mathrm{i}^3} + \dots \\ &= \cos\left(\frac{x}{\mathrm{i}}\right) + \mathrm{i}\sin\left(\frac{x}{\mathrm{i}}\right). \end{aligned}$$

This is better written as

$$\mathrm{e}^{\mathrm{i}x} = \cos(x) + \mathrm{i}\sin(x), \tag{2.5.1}$$

which is the famous *Euler formula.* Then for any complex number $z = x + \mathrm{i}y$,

$$\mathrm{e}^{z} = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^{x}(\cos(y) + \mathrm{i}\sin(y)).$$

Then trigonometric functions and exponential functions can be written in terms of each other:

$$\begin{aligned} \sin(z) &= \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}, \quad \cos(z) = \frac{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2}, \quad \tan(z) = \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{\mathrm{i}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})} \\ \sinh(z) &= \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2}, \quad \cosh(z) = \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2}, \quad \tanh(z) = \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{\mathrm{e}^{z} + \mathrm{e}^{-z}}. \end{aligned}$$

Hence, the following relationships are derived:

$$\sin(z) = -\mathrm{i}\sinh(\mathrm{i}z), \quad \cos(z) = \cosh(\mathrm{i}z), \quad \tan(z) = -\mathrm{i}\tanh(\mathrm{i}z).$$

The complex logarithm, denoted $w = \log(z)$, is the solution to $z = \mathrm{e}^{w}$. We can then define the inverse trigonometric and hyperbolic functions.

We can also define the power function for non-integer powers with $w = z^{\alpha} = \mathrm{e}^{\alpha \log(z)}$. Then power functions can all be written in terms of exponential functions and logarithms. Letting $x = \pi$ in (2.5.1) yields $\mathrm{e}^{\mathrm{i}\pi} = -1$. Furthermore, we can see that exponentiation with an imaginary number is a rotation:

Theorem 2.5.1 *(De Moivre)*: $\forall x \in \mathbb{R}$, $\forall n \in \mathbb{N}$,

$$(\cos(x) + \mathrm{i}\sin(x))^{n} = \cos(nx) + \mathrm{i}\sin(nx).$$

Since all elementary functions can be written in terms of exponential functions and complex logarithms, we will first study the exponential function.

1. The exponential function e^{z} never vanishes as $|\mathrm{e}^{z}| = \mathrm{e}^{x} > 0$.
2. Since $\mathrm{e}^{2\pi\mathrm{i}} = 1$, it is periodic over $2\pi\mathrm{i}$.
3. It is also an entire function with $(\mathrm{e}^{z})' = \mathrm{e}^{z}$.

Write $\mathrm{e}^{z} = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^{x}(\cos(y) + \mathrm{i}\sin(y))$ where $x, y \in \mathbb{R}$. Let $u(x,y) = \mathfrak{Re}(\mathrm{e}^{z}) = \mathrm{e}^{x}\cos(y)$ and $v(x,y) = \mathfrak{Im}(\mathrm{e}^{z}) = \mathrm{e}^{x}\sin(y)$. The first order derivatives are respectively

$$\frac{\partial u}{\partial x} = \mathrm{e}^{x}\cos(y), \quad \frac{\partial u}{\partial y} = -\mathrm{e}^{x}\sin(y),$$

and

$$\frac{\partial v}{\partial x} = \mathrm{e}^{x}\sin(y), \quad \frac{\partial v}{\partial y} = \mathrm{e}^{x}\cos(y),$$

and indeed, the condition described by Theorem 2.2.1 is satisfied.
4 For any two complex numbers z_1 and z_2, $\mathrm{e}^{z_1}\mathrm{e}^{z_2} = \mathrm{e}^{z_1+z_2}$.

In fact, most real exponentiation rules are identical to those in the complex number field. Previously we claimed the periodic properties of e^z. For $U \subseteq \mathbb{C}$, a holomorphic function $f : U \to \mathbb{C}$ is *univalent* over U if it is injective over U. This means that the solutions z_1 and z_2 satisfying $f(z_1) = f(z_2)$ will also always satisfy $z_1 = z_2$.
5 The function e^z is univalent over any horizontal strip of height 2π.

Let $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$, with $x_1, y_1, x_2, y_2 \in \mathbb{R}$, and assume $\mathrm{e}^{z_1} = \mathrm{e}^{z_2}$. Then

$$\mathrm{e}^{x_1}\mathrm{e}^{\mathrm{i}y_1} = \mathrm{e}^{x_2}\mathrm{e}^{\mathrm{i}y_2}.$$

The moduli are equal, and therefore $x_1 = x_2$. By the periodic nature of exponentiation of imaginary numbers, $y_1 - y_2 = 2\pi k$, where $k \in \mathbb{Z}$. To satisfy univalence over a region U, we must exclude distinct points whose imaginary parts differ by an integer multiple of 2π. Thus, we may select U to be any horizontal strip

$$2\pi k \leq \mathfrak{Im}(z) < 2\pi(k+1)$$

or

$$2\pi k < \mathfrak{Im}(z) \leq 2\pi(k+1).$$

Similar to the exponential function, any belt region with thickness 2π is a region over which log is univalent.

Next we examine the complex logarithm.
1 From the periodicity of $z = \mathrm{e}^w$, log is a multi-valued function.
2 Let $z = r\mathrm{e}^{\mathrm{i}\theta}$ and $w = u + \mathrm{i}v$, where $r, \theta, u, v \in \mathbb{R}$. Then

$$r\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{u+\mathrm{i}v},$$

and $\mathrm{e}^u = r$, meaning that $u = \log(r)$ and $v = \theta + 2\pi k$, where $k \in \mathbb{Z}$. Then

$$w = \log(r) + \mathrm{i}(\theta + 2\pi k),$$

and using modulus-argument notation,

$$\log(z) = \log|z| + \mathrm{i}\arg(z),$$

where $\arg(z)$ is the multi-valued argument function. We denote the principal branch of the argument function by

$$\mathrm{Arg} : \mathbb{C} \setminus \{0\} \to (-\pi, \pi].$$

The principal branch of $\log(z)$, or $\operatorname{Log}(z)$, can be defined such that $\mathfrak{Im}(\operatorname{Log}(z)) \in (-\pi, \pi]$.

The functions sin and cos, through their exponential form, still satisfy the familiar properties such as their derivatives, periodicity of 2π, parity, sum and difference formulas, and the fundamental identities

$$\sin^2(z) + \cos^2(z) = 1, \quad \sin(z) = \cos\left(\frac{\pi}{2} - z\right).$$

However, due to the extension, some properties do not hold. For instance, $\sin(z)$ and $\cos(z)$ are unbounded, as along the imaginary axis they resemble their hyperbolic counterparts, which are unbounded along the real line.

We now examine the regions over which they are univalent. Consider

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}.$$

Define the auxiliary functions

$$\xi(z) = iz, \quad \zeta(\xi) = e^{\xi}, \quad w(\zeta) = \frac{\zeta + \frac{1}{\zeta}}{2}.$$

Then

$$\cos(z) = (w \circ \zeta \circ \xi)(z).$$

ξ is clearly univalent on $\mathbb{C}$, as it is a linear map, specifically, a rotation by $\frac{\pi}{2}$ radians followed by scaling. The function ζ is univalent on any domain $U \subseteq \mathbb{C}$ such that for all $\xi_1, \xi_2 \in U$, $\xi_1 - \xi_2 \neq 2\pi i k$ for any $k \in \mathbb{Z}$. If $\xi_1 = iz_1$ and $\xi_2 = iz_2$, then this translates to $z_1 - z_2 \neq 2\pi k$ for $k \in \mathbb{Z}$. The function

$$w(\zeta) = \frac{\zeta + \frac{1}{\zeta}}{2}$$

is univalent on regions excluding pairs (ζ_1, ζ_2) such that $\zeta_1 = \frac{1}{\zeta_2}$. In terms of z, this condition becomes $e^{iz_1} e^{iz_2} \neq 1$, or equivalently, $z_1 + z_2 \neq 2\pi k$ for any $k \in \mathbb{Z}$.

Combining these constraints, we conclude that $\cos(z)$ is univalent on any vertical strip in the complex plane of width π, such as a region of the form

$$\{z \in \mathbb{C} : k\pi < \mathfrak{Re}(z) < (k+1)\pi, k \in \mathbb{Z}\}.$$

Let us now consider the specific region

$$\{z \in \mathbb{C} : 0 < \mathfrak{Re}(z) < \pi\},$$

and analyze how it is mapped under $\cos(z)$.

1 $\xi(z) = \mathrm{i}z$ maps the region $\{z \in \mathbb{C} : 0 < \mathfrak{Re}(z) < \pi\}$ to $\{\xi \in \mathbb{C} : 0 < \mathfrak{Im}(\xi) < \pi\}$.
2 $\zeta(\xi) = \mathrm{e}^{\xi}$ maps this region to the upper half-plane $\mathfrak{Im}(\zeta) > 0$ since $0 < \operatorname{Arg}(\zeta) < \pi$ and $0 < |\zeta|$.
3 $w(\zeta) = \frac{\zeta + \frac{1}{\zeta}}{2}$ maps $\mathfrak{Im}(\zeta) > 0$ to $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$.

Thus, the composition $\cos(z) = w \circ \zeta \circ \xi$ is univalent on the strip

$$\{z \in \mathbb{C} : 0 < \mathfrak{Re}(z) < \pi\},$$

and the image of this strip under cos is

$$\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)).$$

We will now analyze the inverse cosine function, denoted $\arccos(z)$. Consider

$$z = \frac{\mathrm{e}^{\mathrm{i}w} + \mathrm{e}^{-\mathrm{i}w}}{2}.$$

Then

$$\begin{aligned} \left(\mathrm{e}^{\mathrm{i}w}\right)^2 + 1 &= 2z\mathrm{e}^{\mathrm{i}w} \\ \mathrm{e}^{\mathrm{i}w} &= \frac{2z \pm \sqrt{4z^2 - 4}}{2} \\ w &= -\mathrm{i}\log\left(z \pm \sqrt{z^2 - 1}\right). \end{aligned}$$

Then arccos is also a multi-valued function. We can also define

$$\arcsin(z) = \frac{\pi}{2} - \arccos(z).$$

Lastly, we will examine the power function. Let $\alpha = u + \mathrm{i}v$ where $u, v \in \mathbb{R}$. Then

$$z^{\alpha} = \exp(\alpha \log(z)) = \exp((u + \mathrm{i}v)(\log|z| + \mathrm{i}\arg(z))),$$

and in polar form,

$$z^{\alpha} = \exp(u\log|z| - v\arg(z))\exp(\mathrm{i}(v\log|z| + u\arg(z))).$$

Let

$$r_k = \exp(u\log|z| - v\arg(z))$$

and

$$\theta_k = v\log|z| + u\arg(z).$$

Then $z^\alpha = r_k e^{i\theta_k}$, where $k \in \mathbb{Z}$. Analyzing the coefficient of v in the exponent of r_k, z^α is multi-valued if $v \neq 0$.

Then assuming $v = 0$, we have

$$z^\alpha = |z|^u \exp(iu \arg(z)).$$

Doing casework on α,

1 If $\alpha = u \in \mathbb{Z}$, then u can be absorbed into k, and z^α is single-valued.

2 If $\alpha = u \in \mathbb{Q}$ with reduced fractional form $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q > 0$, and $\gcd(p, q) = 1$, then the multi-valued function z^α is given by

$$\begin{aligned} z^\alpha &= |z|^{\frac{p}{q}} \exp\left(i\left(\frac{p}{q}\right)(\operatorname{Arg}(z) + 2\pi k)\right) \\ &= |z|^{\frac{p}{q}} \exp\left(i\left(\frac{p}{q}\right)\operatorname{Arg}(z)\right) \exp\left(2\pi i\left(\frac{p}{q}\right)k\right), \end{aligned}$$

for $k \in \mathbb{Z}$. These values are periodic with period q, since

$$\exp\left(2\pi i\left(\frac{p}{q}\right)(k+q)\right) = \exp\left(2\pi i\left(\frac{p}{q}\right)k\right),$$

as $\exp(2\pi i p) = 1$ for integer p. To prove there are exactly q distinct values, consider $k = 0, 1, 2, ..., q-1$. The exponential factors are $\exp\left(2\pi i\left(\frac{p}{q}\right)k\right)$. These are distinct if, for $0 \leq j < k \leq q-1$,

$$\exp\left(2\pi i\left(\frac{p}{q}\right)j\right) \neq \exp\left(2\pi i\left(\frac{p}{q}\right)k\right),$$

which holds unless $\left(\frac{p}{q}\right)(k-j) \in \mathbb{Z}$, or equivalently unless q divides $p(k-j)$. Since $\gcd(p, q) = 1$, q must divide $k - j$, but $|k-j| < q$ and $k - j \neq 0$, a contradiction. Thus, z^α has exactly q distinct values.

3 If $\alpha = u \in \mathbb{R} \setminus \mathbb{Q}$, then z^α is infinite-valued.

Lastly, there exist series representations of functions using power functions, namely Taylor series, and trigonometric functions, namely Fourier series. There does not exist another distinct representation using exponential functions, as trigonometric functions can be written in terms of them.

3 Complex Integration

3.1 The Cauchy–Goursat Theorem

It is important to know the differential 2-forms even for a single variable complex function. Consider $z = x + iy$ and $\overline{z} = x - iy$. We can then define their corresponding differentials:

$$\mathrm{d}z = \mathrm{d}x + \mathrm{i}\,\mathrm{d}y, \quad \mathrm{d}\overline{z} = \mathrm{d}x - \mathrm{i}\,\mathrm{d}y.$$

The antisymmetric properties of differential forms still hold in complex space. By taking the wedge product of the two basis complex differential forms, we get

$$\begin{aligned}\mathrm{d}\overline{z} \wedge \mathrm{d}z &= (\mathrm{d}x - \mathrm{i}\,\mathrm{d}y) \wedge (\mathrm{d}x + \mathrm{i}\,\mathrm{d}y) \\ &= 2\mathrm{i}\,\mathrm{d}x \wedge \mathrm{d}y.\end{aligned}$$

Analogous to the real case, a 0-form is defined as a scalar-valued function in the form $f(z, \overline{z})$, a 1-form in the form $\omega_0\,\mathrm{d}z + \omega_1\,\mathrm{d}\overline{z}$, and a 2-form as $\omega_0\,\mathrm{d}z \wedge \mathrm{d}\overline{z}$. The exterior differential operator for this one-dimensional case is defined as $\partial + \overline{\partial}$, where $\partial = \mathrm{d}z \wedge \frac{\partial}{\partial z}$ and $\overline{\partial} = \mathrm{d}\overline{z} \wedge \frac{\partial}{\partial \overline{z}}$. Occasionally, one will informally use ∂ and $\overline{\partial}$ as an abbreviation for $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ respectively.

Theorem 3.1.1 *(Lusin Area Theorem)*: For a region $U \subset \mathbb{C}$ and $f : U \to \mathbb{C}$ univalent, the area of the image $f(U)$ is equal to

$$\int_U |f'(z)|^2\,\mathrm{d}A.$$

Proof: We aim to find

$$\int_{f(U)} \mathrm{d}A.$$

By the properties above,

$$\begin{aligned}\int_{f(U)} \mathrm{d}u \wedge \mathrm{d}v &= \frac{\mathrm{i}}{2}\int_{f(U)} \mathrm{d}w \wedge \mathrm{d}\overline{w} = \frac{\mathrm{i}}{2}\int_U \mathrm{d}f(z) \wedge \mathrm{d}\overline{f(z)} \\ &= \frac{\mathrm{i}}{2}\int_U [f'(z)\,\mathrm{d}z] \wedge \left[\overline{f'(z)}\,\mathrm{d}\overline{z}\right] = \int_U |f'(z)|^2\,\mathrm{d}x \wedge \mathrm{d}y,\end{aligned}$$

as desired. □

Remark: The Jacobian determinant of u, v with respect to x, y, for a holomorphic function $f(z) = u(x, y) + \mathrm{i}v(x, y)$ is equal to

$$\begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = |f'(z)|^2$$

by (3.1.6).

Theorem 3.1.2 *(Green's Theorem, Complex Form)*: Let $U \subset \mathbb{C}$ be bounded with a piecewise smooth boundary ∂U. For two scalar functions $\omega_1 =$

$\omega_1(z, \overline{z})$ and $\omega_2 = \omega_2(z, \overline{z})$ satisfying $\omega_1, \omega_2 \in C^1\left(\overline{U}\right)$, define the 1-form $\omega = \omega_1 \, \mathrm{d}z + \omega_2 \, \mathrm{d}\overline{z}$. Then,

$$\int_{\partial U} \omega = \int_U \mathrm{d}\omega. \tag{3.1.1}$$

Proof: For real-valued functions $\xi_1, \xi_2, \eta_1, \eta_2$, let

$$\omega_1 = \xi_1 + \mathrm{i}\eta_1 \quad \text{and let} \quad \omega_2 = \xi_2 + \mathrm{i}\eta_2.$$

Then,

$$\begin{aligned}
\omega &= (\xi_1 + \mathrm{i}\eta_1) \, \mathrm{d}z + (\xi_2 + \mathrm{i}\eta_2) \, \mathrm{d}\overline{z} \\
&= (\xi_1 + \mathrm{i}\eta_1)(\mathrm{d}x + \mathrm{i} \, \mathrm{d}y) + (\xi_2 + \mathrm{i}\eta_2)(\mathrm{d}x - \mathrm{i} \, \mathrm{d}y) \\
&= \xi_1 \, \mathrm{d}x + \mathrm{i}\eta_1 \, \mathrm{d}x + \mathrm{i}\xi_1 \, \mathrm{d}y - \eta_1 \, \mathrm{d}y + \xi_2 \, \mathrm{d}x + \mathrm{i}\eta_2 \, \mathrm{d}x - \mathrm{i}\xi_2 \, \mathrm{d}y + \eta_2 \, \mathrm{d}y \\
&= [(\xi_1 + \xi_2) \, \mathrm{d}x + (\eta_2 - \eta_1) \, \mathrm{d}y] + \mathrm{i}[(\eta_1 + \eta_2) \, \mathrm{d}x + (\xi_1 - \xi_2) \, \mathrm{d}y]
\end{aligned} \tag{3.1.2}$$

Each of $\xi_1, \xi_2, \eta_1, \eta_2$ are real-valued functions that can be represented with a domain of $\mathbb{R}^2$. By definition,

$$\begin{aligned}
\mathrm{d}\omega &= \left(\partial + \overline{\partial}\right)(\xi_1 + \mathrm{i}\eta_1) \, \mathrm{d}z + \left(\partial + \overline{\partial}\right)(\xi_2 + \mathrm{i}\eta_2) \, \mathrm{d}\overline{z} \\
&= \left(\frac{\partial \xi_1}{\partial \overline{z}} + \mathrm{i}\frac{\partial \eta_1}{\partial \overline{z}}\right) \mathrm{d}\overline{z} \wedge \mathrm{d}z + \left(\frac{\partial \xi_2}{\partial z} + \mathrm{i}\frac{\partial \eta_2}{\partial z}\right) \mathrm{d}z \wedge \mathrm{d}\overline{z} \\
&= 2\left(\mathrm{i}\frac{\partial \xi_1}{\partial \overline{z}} - \frac{\partial \eta_1}{\partial \overline{z}} - \mathrm{i}\frac{\partial \xi_2}{\partial z} + \frac{\partial \eta_2}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\mathrm{i}\frac{\partial \xi_1}{\partial x} - \frac{\partial \xi_1}{\partial y} - \frac{\partial \eta_1}{\partial x} - \mathrm{i}\frac{\partial \eta_1}{\partial y} - \mathrm{i}\frac{\partial \xi_2}{\partial x} - \frac{\partial \xi_2}{\partial y} + \frac{\partial \eta_2}{\partial x} - \mathrm{i}\frac{\partial \eta_2}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial \eta_2}{\partial x} - \frac{\partial \xi_1}{\partial y} - \frac{\partial \eta_1}{\partial x} - \frac{\partial \xi_2}{\partial y}\right) \mathrm{d}A + \mathrm{i}\left(\frac{\partial \xi_1}{\partial x} - \frac{\partial \eta_1}{\partial y} - \frac{\partial \xi_2}{\partial x} - \frac{\partial \eta_2}{\partial y}\right) \mathrm{d}A.
\end{aligned} \tag{3.1.3}$$

From (3.1.2), we can apply Theorem 1.2.8. For the real component of ω, we obtain

$$\oint_{\partial U} (\xi_1 + \xi_2) \, \mathrm{d}x + (\eta_2 - \eta_1) \, \mathrm{d}y = \iint_U \left(\frac{\partial \eta_2}{\partial x} - \frac{\partial \xi_1}{\partial y} - \frac{\partial \eta_1}{\partial x} - \frac{\partial \xi_2}{\partial y}\right) \mathrm{d}x \, \mathrm{d}y,$$

and for the imaginary component,

$$\oint_{\partial U} (\eta_1 + \eta_2) \, \mathrm{d}x + (\xi_1 - \xi_2) \, \mathrm{d}y = \iint_U \left(\frac{\partial \xi_1}{\partial x} - \frac{\partial \eta_1}{\partial y} - \frac{\partial \xi_2}{\partial x} - \frac{\partial \eta_2}{\partial y}\right) \mathrm{d}x \, \mathrm{d}y,$$

and the integrands on the right side both match those of (3.1.3). ☐

The theorem above is only a specific case of the Stokes–Cartan Theorem (Theorem 1.2.11). However, it proves the validity of the treatment of the ∂ and $\overline{\partial}$ operators, and the generalization to forms with basis $\mathrm{d}z$ and $\mathrm{d}\overline{z}$.

Theorem 3.1.3 *(Cauchy–Pompeiu)*: Let $U \subset \mathbb{C}$ be bounded with a piecewise C^1 boundary ∂U. Let $f(z) \in C^1\left(\overline{U}\right)$. Then $\forall z \in U \setminus \partial U$,

$$f(z) = \frac{1}{2\pi \mathrm{i}}\left(\oint_{\partial U} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \int_U \frac{\partial f(\zeta)}{\partial \overline{\zeta}} \frac{\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta}{\zeta - z}\right).$$

Proof: Since $z \in U \setminus \partial U$, $\exists \varepsilon > 0$ such that $D(z, \varepsilon) \subset U$. Consider the complex differential form

$$\frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}$$

with a singularity at $\zeta = z$. Consider the region $U \setminus D(z, \varepsilon)$. Since $f \in C^1\left(\overline{U}\right)$, by applying Green's Theorem (Theorem 3.1.2),

$$\int_{U \setminus D(z,\varepsilon)} \mathrm{d}\left(\frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}\right) = \oint_{\partial U} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - \oint_{\partial D(z,\varepsilon)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}. \qquad (3.1.4)$$

By properties of d, the expression is equal to

$$\begin{aligned}\int_{U \setminus D(z,\varepsilon)} \left(\partial + \overline{\partial}\right)\left(\frac{f(\zeta)}{\zeta - z}\right) \wedge \mathrm{d}\zeta = \int_{U \setminus D(z,\varepsilon)} & \frac{\partial}{\partial \zeta}\left(\frac{f(\zeta)}{\zeta - z}\right) \mathrm{d}\zeta \wedge \mathrm{d}\zeta \\ & + \frac{\partial}{\partial \overline{\zeta}}\left(\frac{f(\zeta)}{\zeta - z}\right) \mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta.\end{aligned}$$

The first term in the integrand vanishes as it contains $\mathrm{d}\zeta \wedge \mathrm{d}\zeta$. The second term can be simplified using the fact that $\frac{\partial \frac{1}{\zeta - z}}{\partial \overline{\zeta}} = 0$, leading to

$$\int_{U \setminus D(z,\varepsilon)} \frac{\partial f}{\partial \overline{\zeta}} \cdot \frac{\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta}{\zeta - z}.$$

The rightmost term in (3.1.4) can be parameterized with $\zeta = z + \varepsilon \mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$. Then,

$$\begin{aligned}
\oint_{\partial D(z,\varepsilon)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} &= \int_0^{2\pi} \frac{f\left(z + \varepsilon \mathrm{e}^{\mathrm{i}t}\right)}{\varepsilon \mathrm{e}^{\mathrm{i}t}} \cdot \mathrm{i}\varepsilon \mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t \\
&= \mathrm{i} \int_0^{2\pi} f\left(z + \varepsilon \mathrm{e}^{\mathrm{i}t}\right) \mathrm{d}t \\
&= \mathrm{i} \int_0^{2\pi} \left(f\left(z + \varepsilon \mathrm{e}^{\mathrm{i}t}\right) - f(z)\right) \mathrm{d}t + \mathrm{i} \int_0^{2\pi} f(z)\,\mathrm{d}t.
\end{aligned}$$

Because $f \in C^1\left(\overline{U}\right)$, by Proposition 1.2.2, f is Lipschitz continuous on $\overline{U}$, and $\exists M \in \mathbb{R}_{>0}$ such that $\forall z_0, z_1 \in \overline{U}$, $|f(z_1) - f(z_0)| \leq M|z_1 - z_0|$. On $\partial D(z, \varepsilon)$, we get that $\left|f\left(z + \varepsilon \mathrm{e}^{\mathrm{i}t}\right) - f(z)\right| \leq M\varepsilon$. Therefore,

$$\left|\int_0^{2\pi} \left(f\left(z + \varepsilon \mathrm{e}^{\mathrm{i}t}\right) - f(z)\right) \mathrm{d}t\right| \leq \int_0^{2\pi} \left|f\left(z + \varepsilon \mathrm{e}^{\mathrm{i}t}\right) - f(z)\right| \mathrm{d}t \leq 2M\pi\varepsilon,$$

which approaches 0 as $\varepsilon \to 0$. Taking this limit, we obtain

$$2\pi\mathrm{i}f(z) = \oint_{\partial U} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - \int_U \frac{\partial f}{\partial \overline{\zeta}} \cdot \frac{\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta}{\zeta - z} + \lim_{\varepsilon \to 0} \int_{D(z,\varepsilon)} \frac{\partial f}{\partial \overline{\zeta}} \cdot \frac{\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta}{\zeta - z} \qquad (3.1.5)$$

We then aim to prove that

$$\lim_{\varepsilon \to 0} \int_{D(z,\varepsilon)} \frac{\partial f}{\partial \overline{\zeta}} \cdot \frac{\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta}{\zeta - z} = 0. \qquad (3.1.6)$$

Notice that since $f \in C^1\left(\overline{U}\right)$, by Theorem 1.2.13, $\exists M' \in \mathbb{R}_{>0}$ such that $\forall \zeta \in \overline{U}$, $\left|\frac{\partial f}{\partial \overline{\zeta}}\right| \leq M'$. Then,

$$\lim_{\varepsilon \to 0} \left|\int_{D(z,\varepsilon)} \frac{\partial f}{\partial \overline{\zeta}} \cdot \frac{\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta}{\zeta - z}\right| \leq M' \lim_{\varepsilon \to 0} \left|\int_{D(z,\varepsilon)} \frac{1}{\zeta - z}\,\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta\right|.$$

By a change of variables to a polar coordinate system centered at z, we obtain

$$M' \lim_{\varepsilon \to 0} \left|\int_{D(z,\varepsilon)} \frac{1}{\mathrm{r}\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{d}\left(z + r\mathrm{e}^{-\mathrm{i}\theta}\right) \wedge \mathrm{d}\left(z + r\mathrm{e}^{\mathrm{i}\theta}\right)\right|,$$

and by expansion of the wedge product,

$$\begin{aligned} M' \lim_{\varepsilon\to 0}\left|\int_{D(z,\varepsilon)} \frac{2\mathrm{i}}{\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{d}r\wedge\mathrm{d}\theta\right| &= 2M' \lim_{\varepsilon\to 0}\left|\int_{D(z,\varepsilon)} \frac{1}{\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{d}r\wedge\mathrm{d}\theta\right| \\ &= 2M' \lim_{\varepsilon\to 0}\left|\int_0^{2\pi}\int_0^{\varepsilon} \mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}r\,\mathrm{d}\theta\right| \\ &= 0. \end{aligned}$$

Then from rearranging (3.1.5), we obtain:

$$f(z) = \frac{1}{2\pi\mathrm{i}}\left(\oint_{\partial U} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - \int_U \frac{\partial f}{\partial\overline{\zeta}}\cdot\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{\zeta - z}\right). \qquad \square$$

Corollary 3.1.3.1: Let $f : \mathbb{C} \to \mathbb{C}$ be a continuously differentiable, compactly supported function. Then

$$f(z) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{\partial f}{\partial\overline{\zeta}}\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{\zeta - z}$$

for all $z \in \mathbb{C}$ where $\zeta = \xi + \mathrm{i}\eta$.

Proof: Choose $R > 0$ such that $D(0, R) \supset \operatorname{supp}(f)$. By the Cauchy–Pompeiu Theorem (Theorem 3.1.3), we have

$$f(z) = \frac{1}{\pi}\left(\frac{1}{2\mathrm{i}}\oint_{\partial D(0,R)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - \iint_{D(0,R)} \frac{\partial f}{\partial\overline{\zeta}}\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{\zeta - z}\right).$$

Then the proof is complete given that f vanishes on $\partial D(0, R)$ and by letting $R \to \infty$. □

In complex analysis, when integrating over a region that contains a singularity, it is common to exclude a small disk of radius ε around the singularity, perform the integration over the punctured region, and then take the limit as $\varepsilon \to 0$. As in the proof above, the displayed estimate for the integral over the removed disk is still necessary in confirmation, although it is typically tacitly elided.

From the above result, we can directly obtain the following theorem:

Theorem 3.1.4 *(Cauchy's Integral Formula)*: Let $U \subset \mathbb{C}$ be an open region with a piecewise C^1 boundary ∂U, and let $f \in C^1\left(\overline{U}\right)$ be holomorphic on U. Then for all $z \in U$,

$$f(z) = \frac{1}{2\pi\mathrm{i}}\oint_{\partial U} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta. \tag{3.1.7}$$

Proof: By (3.1.5), for $f\left(\zeta, \overline{\zeta}\right)$, $\frac{\partial f}{\partial \overline{\zeta}} = 0$. Applying the Cauchy–Pompeiu Theorem (Theorem 3.1.3), the area integral vanishes, and (3.1.7) consequently follows. □

Theorem 3.1.5 *(Cauchy's Integral Theorem)*: Let $U \subset \mathbb{C}$ be an open region with piecewise C^1 boundary ∂U. For a function $f(z) \in C^1\left(\overline{U}\right)$ holomorphic over U,

$$\oint_{\partial U} f(\zeta) \,\mathrm{d}\zeta = 0.$$

Proof: Let $\psi(z) = z f(z)$. Applying Theorem 3.1.4 on $\psi(\zeta)$ with $z = 0$, we obtain

$$0 = \frac{1}{2\pi \mathrm{i}} \oint_{\partial U} \frac{\psi(\zeta)}{\zeta} \,\mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \oint_{\partial U} f(\zeta) \,\mathrm{d}\zeta.$$

Alternatively, we can use Green's Theorem (Theorem 3.1.2) with $\omega = f(\zeta)\,\mathrm{d}\zeta$:

$$\oint_{\partial U} f(\zeta)\,\mathrm{d}\zeta = \oint_{\partial U} \omega = \int_U \mathrm{d}\omega = \int_U \frac{\partial f}{\partial \overline{\zeta}} \,\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta = 0.$$ □

Theorem 3.1.6: For a compactly supported function $\psi(z) \in C^1(\mathbb{C})$, a solution satisfying $u(z) \in C^1(\mathbb{C})$ to the non-homogeneous Cauchy–Riemann equation

$$\frac{\partial u(z)}{\partial \overline{z}} = \psi(z)$$

is

$$u(z) = -\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} \frac{\psi(\zeta)}{\zeta - z} \,\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta. \tag{3.1.8}$$

Proof: Split $\mathbb{C}$ into $\mathbb{C} \setminus D(z, \varepsilon)$ and $\overline{D(z, \varepsilon)}$. For all $\varepsilon > 0$, the integral

$$-\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C} \setminus D(z,\varepsilon)} \frac{\psi(\zeta)}{\zeta - z} \,\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta$$

is continuous. Since $\psi(\zeta)$ is compactly supported over $\mathbb{C}$ and continuous, by Theorem 1.2.13, it is bounded. Then the limit

$$\lim_{\varepsilon \to 0} \left(-\frac{1}{2\pi \mathrm{i}} \int_{D(z,\varepsilon)} \frac{\psi(\zeta)}{\zeta - z} \,\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta \right) = 0.$$

Therefore, (3.1.8) is continuous. A trivial substitution can be used to rewrite

$$u(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} \frac{\psi(\zeta + z)}{\zeta} \, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$$

Then,

$$\frac{u(z + \Delta z) - u(z)}{\Delta z} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} \frac{\psi(\zeta + z + \Delta z) - \psi(\zeta + z)}{\Delta z \zeta} \, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \tag{3.1.9}$$

For a fixed z, the value of

$$\frac{\psi(\zeta + z + \Delta z) - \psi(\zeta + z)}{\Delta z}$$

tends to $\frac{\partial \psi(\zeta + z)}{\partial \zeta}$ as $\Delta z \to 0$. Because $\psi(\zeta) = \psi(\zeta + z)$ has compact support and is C^1, by Proposition 1.2.2, it is Lipschitz continuous for a constant M. Let $|\Delta z| < 1$ and let $K = \left\{ w \in \mathbb{C} : \inf_{\zeta \in \operatorname{supp} \phi} |w - \zeta| \leq 1 \right\}$. Then,

$$\left| \frac{\psi(\zeta + z + \Delta z) - \psi(\zeta + z)}{\Delta z} \right| \leq M,$$

and specifically, when $\zeta + z \notin K$,

$$\frac{\psi(\zeta + z + \Delta z) - \psi(\zeta + z)}{\Delta z} = 0.$$

As shown above, the integrand is uniformly bounded by M, which has a convergent integral of $\int_K M \, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$, the limit $\Delta z \to 0$ may commute with the integral in (3.1.9). Let $\zeta = \xi + \mathrm{i}\eta$. From the real axis,

$$\frac{\partial u}{\partial x}(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} \frac{1}{\zeta} \frac{\partial \psi}{\partial \xi}(\zeta + z) \, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} \frac{\partial \psi(\zeta)}{\partial \xi} \cdot \frac{1}{\zeta - z} \, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \tag{3.1.10}$$

From the imaginary axis,

$$\frac{\partial u}{\partial y}(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} \frac{1}{\zeta} \frac{\partial \psi}{\partial \eta}(\zeta + z) \, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} \frac{\partial \psi(\zeta)}{\partial \eta} \cdot \frac{1}{\zeta - z} \, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \tag{3.1.11}$$

Since $\psi \in C^1(\mathbb{C})$ and has Lipschitz constant M, (3.1.10), (3.1.11) are both continuous (by the same argument for the continuity of $u(z)$). Thus, $u \in C^1(\mathbb{C})$. It follows from the two equations that

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} \frac{\partial \psi}{\partial \bar{\zeta}} \cdot \frac{1}{\zeta - z} \, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} = \frac{1}{2\pi \mathrm{i}} \int_{K} \frac{\partial \psi}{\partial \bar{\zeta}} \cdot \frac{1}{\zeta - z} \, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.$$

By Corollary 3.1.3.1,

$$\frac{\partial u}{\partial \bar{z}} = \psi(z). \qquad \square$$

Remark: In the first part, we established that a function $\psi(z) \in C^0(\mathbb{C})$ with compact support satisfies

$$u(z) = -\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{C}} \frac{\psi(\zeta)}{\zeta - z} \,\mathrm{d}\bar{\zeta} \wedge \mathrm{d}\zeta \in C^0(\mathbb{C}).$$

If $\psi(z) \in C^1(\mathbb{C})$, then the first order derivatives of $u(z)$ can be written in the same form ((3.1.10), (3.1.11)) since $\frac{\partial\psi}{\partial\xi}, \frac{\partial\psi}{\partial\eta} \in C^0(\mathbb{C})$ and are also compactly supported. Then they too are continuous functions, and $u(z) \in C^1(\mathbb{C})$.

Then using the same argument, In general, for $\psi(z) \in C^k(\mathbb{C})$, the same process can be used recursively to find that $u(z) \in C^k(\mathbb{C})$ as well.

If the support of $\psi(z)$ is the union of infinitely many or finitely many disjoint compact sets, then the integral in (3.1.8) can be split into a sum of integrals over each compact set, and the same argument applies to each term.

When Cauchy formalized Theorem 3.1.4, Theorem 3.1.5, he included the necessary condition that $f(z) \in C^1\left(\overline{U}\right)$. It was later shown that all such holomorphic functions had holomorphic derivatives, and this condition was thus later dropped by Goursat:

Lemma 3.1.1: Let $f : G \to \mathbb{C}$ be a continuous function defined for a region $G \subseteq \mathbb{C}$. Let $\Gamma \subset G$ be a rectifiable piecewise smooth curve. Then $\forall \varepsilon > 0$, there exists a polygonal chain $P \subset G$ inscribing Γ (each vertex lies on Γ) where

$$\left| \int_\Gamma f(z) \,\mathrm{d}z - \int_P f(z) \,\mathrm{d}z \right| < \varepsilon.$$

Proof: Because $f \in C^0(G)$, there is a compact set $D \subseteq G$ enclosing Γ and is the closure of some open set. By Theorem 1.2.15, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall z', z'' \in D$ satisfying $|z'' - z'| < \delta$, $|f(z'') - f(z')| < \varepsilon$. Partition Γ into $n \in \mathbb{N}$ curves $\gamma_0, \gamma_1, ..., \gamma_{n-1}$ between points $z_0, z_1, ...z_n$ such that $\forall k \in \{0, 1, ..., n-1\}$ the length of γ_k is less than δ. For all $k \in \{0, 1, ..., n-1\}$, let l_k denote the straight line segment connecting z_k and z_{k+1}. The length of l_k is less than δ as well. Then let

$$P = \bigcup_{k=0}^{n-1} l_k.$$

Over the partition formed with γ_k, the integral

$$\int_{\Gamma} f(z)\,\mathrm{d}z$$

can be approximated with the Riemann sum

$$S = \sum_{k=0}^{n-1} f(z_k)\Delta z_k$$

where

$$\Delta z_k = z_{k+1} - z_k = \int_{\gamma_k} \mathrm{d}z = \int_{l_k} \mathrm{d}z.$$

Then the sum above can be written as

$$S = \sum_{k=0}^{n-1} \int_{\gamma_k} f(z_k)\,\mathrm{d}z = \sum_{k=0}^{n-1} \int_{l_k} f(z_k)\,\mathrm{d}z,$$

and it follows that

$$\left| \int_{\Gamma} f(z)\,\mathrm{d}z - S \right| = \left| \sum_{k=0}^{n-1} \int_{\gamma_k} [f(z) - f(z_k)]\,\mathrm{d}z \right| < \varepsilon \cdot \mathrm{length}(\Gamma)$$

and

$$\left| \int_{P} f(z)\,\mathrm{d}z - S \right| = \left| \sum_{k=0}^{n-1} \int_{l_k} [f(z) - f(z_k)]\,\mathrm{d}z \right| < \varepsilon \cdot \mathrm{length}(P) < \varepsilon \cdot \mathrm{length}(\Gamma)$$

where $\mathrm{length}(\Gamma)$ is the length of Γ and $\mathrm{length}(P)$ is the length of P. Then,

$$\begin{aligned} \left| \int_{\Gamma} f(z)\,\mathrm{d}z - \int_{P} f(z)\,\mathrm{d}z \right| &\leq \left| \int_{\Gamma} f(z)\,\mathrm{d}z - S \right| + \left| \int_{P} f(z)\,\mathrm{d}z - S \right| \\ &\leq 2\varepsilon \cdot \mathrm{length}(\Gamma). \end{aligned} \qquad \square$$

Lemma 3.1.2 *(Goursat)*: Given a holomorphic function $f(z)$ on a simply connected region $U \subseteq \mathbb{C}$, for any piecewise C^1 closed curve $\Gamma \subset U$,

$$\int_{\Gamma} f(\zeta)\,\mathrm{d}\zeta = 0. \tag{3.1.12}$$

Proof: By Lemma 3.1.1, $\forall \varepsilon > 0$, there is a polygonal chain P where

$$\left| \int_{\Gamma} f(z)\,\mathrm{d}z - \int_{P} f(z)\,\mathrm{d}z \right| < \varepsilon. \tag{3.1.13}$$

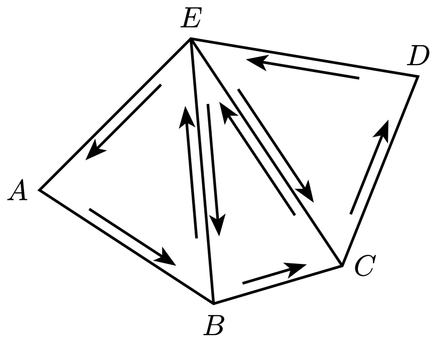

Figure 1: Closed triangulated polygonal chain

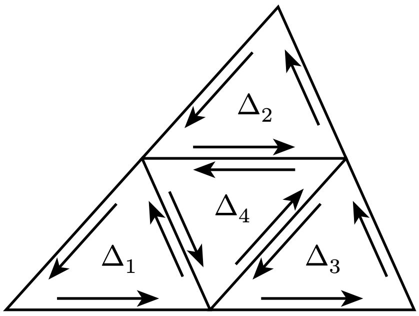

Figure 2: Quadrisection of int Δ

The statement we aim to prove is equivalent to proving that

$$\int_P f(z)\,\mathrm{d}z = 0. \tag{3.1.14}$$

Since P is a closed polygonal chain, we can triangulate the interior. For example, consider Figure 1, where

$$\begin{aligned}\oint_{\pentagon ABCDE} f(z)\,\mathrm{d}z &= \left(\int_{\overrightarrow{AB}} + \int_{\overrightarrow{BC}} + \int_{\overrightarrow{CD}} + \int_{\overrightarrow{DE}} + \int_{\overrightarrow{EA}}\right) f(z)\,\mathrm{d}z \\ &\quad + \left(\int_{\overrightarrow{BE}} + \int_{\overrightarrow{EB}} + \int_{\overrightarrow{CE}} + \int_{\overrightarrow{EC}}\right) f(z)\,\mathrm{d}z \\ &= \oint_{\Delta ABE} f(z)\,\mathrm{d}z + \oint_{\Delta BCE} f(z)\,\mathrm{d}z + \oint_{\Delta CDE} f(z)\,\mathrm{d}z.\end{aligned}$$

Thus, if the integral over every triangle in U vanishes, then (3.1.12) follows. Consider a triangle in U with boundary Δ. Then define M to be

$$M = \left|\oint_{\Delta} f(z)\,\mathrm{d}z\right|.$$

We can quadrisect the triangle bounded by Δ into four triangles with boundaries $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ as in Figure 2. Then one of Δ_1, Δ_2, Δ_3, or Δ_4 (denote this to be Δ^1) satisfy

$$\left|\oint_{\Delta^1} f(z)\,\mathrm{d}z\right| \geq \frac{M}{4},$$

and recursively, choose

$$\left|\oint_{\Delta^2} f(z)\,\mathrm{d}z\right| \geq \frac{M}{4^2}, \ldots, \left|\oint_{\Delta^n} f(z)\,\mathrm{d}z\right| \geq \frac{M}{4^n}. \tag{3.1.15}$$

Let L denote the perimeter of Δ. Then, the perimeters of $\Delta^1, \Delta^2, \ldots$ respectively are $\frac{L}{2}, \frac{L}{2^2}, \ldots$. As $n \to \infty$, Δ_n shrinks to a single point z_0. Then, $\forall n \in \mathbb{N}$, $z_0 \in \Delta^n$.

By the definition of holomorphy, $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall z \in D(z_0, \delta)$,

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon,$$

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon|z - z_0|,$$

and $\exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}_{>N}$, $\Delta^n \subset D(z_0, \delta)$. By Theorem 3.1.5, since the functions $z \to 1$ and $z \to z$ are both entire,

$$\oint_{\Delta^n} \mathrm{d}z = 0, \oint_{\Delta^n} z\,\mathrm{d}z = 0.$$

Then

$$\begin{aligned}\oint_{\Delta^n} f(z)\,\mathrm{d}z &= \oint_{\Delta^n} f(z)\,\mathrm{d}z - f(z_0)\oint_{\Delta^n} \mathrm{d}z - f'(z_0)\left(\oint_{\Delta^n} z\,\mathrm{d}z - z_0\oint_{\Delta^n} \mathrm{d}z\right) \\ &= \oint_{\Delta^n} [f(z) - f(z_0) - f'(z_0)(z - z_0)]\,\mathrm{d}z.\end{aligned}$$

Because the distance between any two points in the interior of a triangle is always less than its perimeter, using the triangle inequality for complex integrals,

$$\begin{aligned}\left|\oint_{\Delta^n} [f(z) - f(z_0) - f'(z_0)(z - z_0)]\,\mathrm{d}z\right| &\leq \varepsilon \oint_{\Delta^n} |z - z_0||\mathrm{d}z| \\ &= \frac{\varepsilon L}{2^n}\oint_{\Delta^n} |\mathrm{d}z| = \frac{\varepsilon L^2}{4^n}.\end{aligned}$$

Comparing the above equation with (3.1.15),

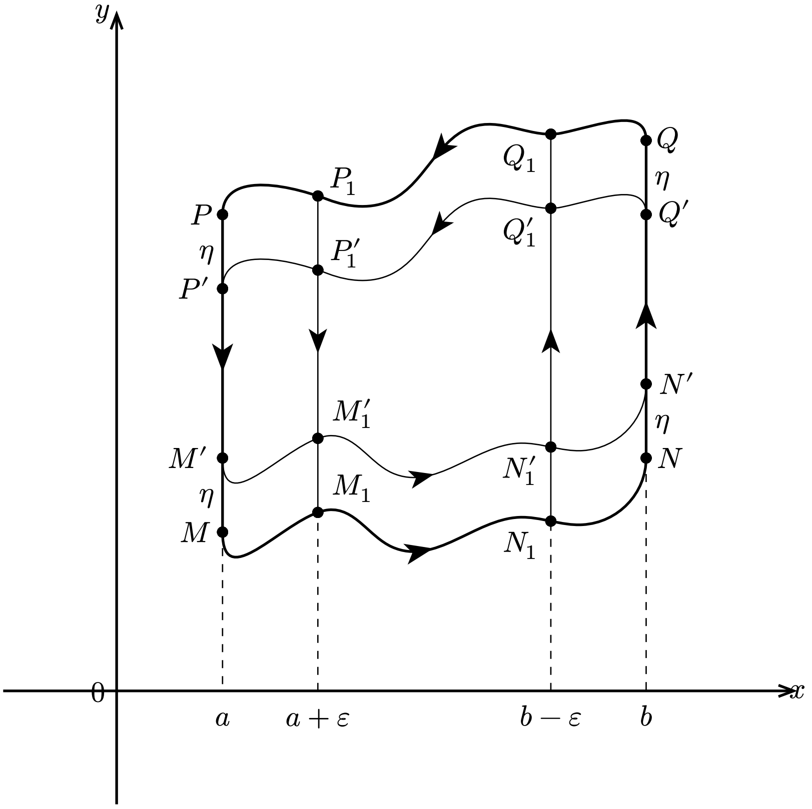

Figure 3: A simplified region containing two vertical lines and two continuous, rectifiable curves.

$$\frac{M}{4^n} < \varepsilon \frac{L}{4^n}, \quad M < \varepsilon L.$$

Since Δ is rectifiable, L is finite, and letting $\varepsilon \to 0$, we find that $M \to 0$. Then, for every triangle in U, the integral vanishes, and (3.1.14), (3.1.13) follow. $\square$

Theorem 3.1.7 *(Cauchy–Goursat)*: Let $U \subset \mathbb{C}$ be an open region bounded with boundary ∂U. Let $f : U \to \mathbb{C}$ be a holomorphic function continuous on $\overline{U}$. Then,

$$\oint_{\partial U} f(\zeta)\, \mathrm{d}\zeta = 0.$$

Proof: Since $\partial U \cap U = \varnothing$ and $f(z)$ is not necessarily holomorphic over $\overline{U}$, we cannot directly apply Lemma 3.1.2.

First assume U has the shape of $MNQP$ in Figure 3. That is, U consists of $x = a$, $x = b$ for $a < b$, and two rectifiable C^0 curves $\overrightarrow{MN} : y = \phi(x)$ and $\overrightarrow{QP} : y = \psi(x)$ such that $\phi(x) < \psi(x)$, $\forall a \leq x \leq b$.

For some $\varepsilon > 0$, $\eta > 0$, construct a new curve $M_1'N_1'Q_1'P_1' \in U$ to be the boundary of the region bounded by $P_1M_1 : x = a + \varepsilon$, $N_1Q_1 : x = b - \varepsilon$, $M'N' : \phi(x) + \eta$, and $Q'P' : \psi(x) - \eta$ such that $M_1'N_1'Q_1'P_1'$ remains simple. By Lemma 3.1.2,

$$\oint_{M_1'N_1'Q_1'P_1'} f(z)\,\mathrm{d}z = 0.$$

By Theorem 1.2.15, $f(z)$ is uniformly continuous over $\overline{U}$, and therefore $\forall \varepsilon' > 0$, we can choose $\eta > 0$ so that $\forall z \in \overrightarrow{M_1'N_1'}$, $|f(z) - f(z - \eta)| < \varepsilon'$ is satisfied. Letting $\eta \to 0$ with $\varepsilon' \to 0$ and fixing $\varepsilon > 0$, we get that

$$\begin{aligned} \left| \int_{\overrightarrow{M_1'N_1'}} f(z)\,\mathrm{d}z - \int_{\overrightarrow{M_1N_1}} f(z)\,\mathrm{d}z \right| &\leq \int_{\overrightarrow{M_1'N_1'}} |f(z) - f(z - \eta)||\mathrm{d}z| \\ &< \varepsilon' \int_{\overrightarrow{M_1'N_1'}} |\mathrm{d}z| \to 0, \end{aligned}$$

and consequently,

$$\int_{\overrightarrow{M_1'N_1'}} f(z)\,\mathrm{d}z \to \int_{\overrightarrow{M_1N_1}} f(z)\,\mathrm{d}z. \tag{3.1.16}$$

Under the same limit, we get

$$\int_{\overrightarrow{Q_1'P_1'}} f(z)\,\mathrm{d}z \to \int_{\overrightarrow{Q_1P_1}} f(z)\,\mathrm{d}z. \tag{3.1.17}$$

By the continuity of $f(z)$ over a compact set,

$$\int_{\overrightarrow{P_1'M_1'}} f(z)\,\mathrm{d}z \to \int_{\overrightarrow{P_1M_1}} f(z)\,\mathrm{d}z, \int_{\overrightarrow{N_1'Q_1'}} f(z)\,\mathrm{d}z \to \int_{\overrightarrow{N_1Q_1}} f(z)\,\mathrm{d}z. \tag{3.1.18}$$

Then letting $\varepsilon \to 0$, for the same reason as (3.1.18), (3.1.16), (3.1.17) yield

$$\int_{\overrightarrow{M_1N_1}} f(z)\,\mathrm{d}z \to \int_{\overrightarrow{MN}} f(z)\,\mathrm{d}z, \int_{\overrightarrow{Q_1P_1}} f(z)\,\mathrm{d}z \to \int_{\overrightarrow{QP}} f(z)\,\mathrm{d}z.$$

We are left to show the subsequent limits of the results from (3.1.18). For the left integral, let $y_\phi = \max(\phi(a), \phi(a + \varepsilon))$ and $y_\psi = \max(\psi(a), \psi(a + \varepsilon))$.

Then,

$$\int_{\overrightarrow{PM}} f(z)\,\mathrm{d}z = \mathrm{i}\int_{\psi(a)}^{\phi(a)} f(a+\mathrm{i}y)\,\mathrm{d}y = \mathrm{i}\left(\int_{\psi(a)}^{y_\phi} + \int_{y_\phi}^{y_\psi} + \int_{y_\psi}^{\phi(a)}\right) f(a+\mathrm{i}y)\,\mathrm{d}y.$$

Similarly,

$$\int_{\overrightarrow{P_1M_1}} f(z)\,\mathrm{d}z = \mathrm{i}\left(\int_{\psi(a+\varepsilon)}^{y_\phi} + \int_{y_\phi}^{y_\psi} + \int_{y_\psi}^{\phi(a+\varepsilon)}\right) f(a+\varepsilon+\mathrm{i}y)\,\mathrm{d}y.$$

The difference $\left(\int_{\overrightarrow{PM}} - \int_{\overrightarrow{P_1M_1}}\right) f(z)\,\mathrm{d}z$ between the two is then equal to

$$\mathrm{i}\int_{y_\phi}^{y_\psi} (f(a+\mathrm{i}y) - f(a+\varepsilon+\mathrm{i}y))\,\mathrm{d}y + \mathrm{i}\left(\int_{\psi(a)}^{y_\phi} + \int_{y_\psi}^{\phi(a)}\right) f(a+\mathrm{i}y)$$
$$-\,\mathrm{i}\left(\int_{\psi(a+\varepsilon)}^{y_\phi} + \int_{y_\psi}^{\phi(a+\varepsilon)}\right) f(a+\varepsilon+\mathrm{i}y).$$

The first term vanishes by uniform continuity, through the same argument used for $M_1'N_1' \to M_1N_1$, and the remaining four integrals all tend to 0 because they are taken over degenerating intervals. As $\varepsilon \to 0, y_\phi \to \phi(a)$ and $y_\psi \to \psi(a)$ because $\phi, \psi \in C^0$. Therefore,

$$\int_{\overrightarrow{P_1M_1}} f(z)\,\mathrm{d}z \to \int_{\overrightarrow{PM}} f(z)\,\mathrm{d}z,$$

and through similar logic,

$$\int_{\overrightarrow{N_1Q_1}} f(z)\,\mathrm{d}z \to \int_{\overrightarrow{NQ}} f(z)\,\mathrm{d}z.$$

Therefore,

$$\oint_{MNQP} f(z)\,\mathrm{d}z = 0.$$

Any open region $U \subset \mathbb{C}$ with a simple closed boundary can be broken up into smaller regions with the same form as $MNQP$ with finitely many auxiliary lines. Then the conclusion follows. ☐

Remark: The theorem is also valid for any multiply connected region (and its boundary will consist of multiple curves) as a multiply connected region is equal to the union of several simply connected regions with vertical auxiliary lines between.

Additionally, if $U \subset \mathbb{C}$ is simply connected and f is holomorphic on U, then for any two points $z, z_0 \in U$, the integral

$$\int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta$$

is well-defined and independent of the path taken from z_0 to z. In this sense, holomorphic functions behave analogously to potential fields.

Theorem 3.1.8 *(Cauchy–Goursat)*: Let $U \subset \mathbb{C}$ be an open region bounded with a simple closed boundary ∂U, and let $f : U \to \mathbb{C}$ be a holomorphic function continuous on $\overline{U}$. Then for all $z \in U$,

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta. \tag{3.1.19}$$

Proof: By the Cauchy–Goursat Theorem (Theorem 3.1.7),

$$\int_{\partial(U \setminus D(z,\varepsilon))} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \oint_{\partial U} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \oint_{\partial D(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 0.$$

From rearrangement,

$$\oint_{\partial U} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 2\pi\mathrm{i} f(z) + \mathrm{i} \int_0^{2\pi} \left[f(z + \varepsilon \mathrm{e}^{\mathrm{i}t}) - f(z)\right] \mathrm{d}t.$$

Since $f \in C^0(\partial D(z, \varepsilon))$, as $\varepsilon \to 0$,

$$\begin{aligned} \left| \int_0^{2\pi} \left[f(z + \varepsilon \mathrm{e}^{\mathrm{i}t}) - f(z)\right] \mathrm{d}t \right| &\leq \int_0^{2\pi} \left| f(z + \varepsilon \mathrm{e}^{\mathrm{i}t}) - f(z) \right| \mathrm{d}t \\ &\leq 2\pi \max_{t \in [0, 2\pi]} \left| f(z + \varepsilon \mathrm{e}^{\mathrm{i}t}) - f(z) \right| \to 0. \end{aligned}$$

By rearrangement,

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

□

Remark: In the proof of Theorem 3.1.3, we used Lipschitz continuity for a smooth function, which was a stronger condition than necessary. The true necessity of smoothness was to be able to apply Green's Theorem (Theorem 3.1.2).

This profound theorem is extremely important and helpful in complex integration and essential in the evaluation of integrals, as demonstrated below.

Example 3.1.1: Evaluate the integral $\oint_{\partial D(0,2)} \frac{\mathrm{d}z}{z^n-1}$, where $n \in \mathbb{N}_{\geq 2}$.

Proof: Since $z^n - 1 = \prod_{k=0}^{n-1}\left(z - \omega_n^k\right)$, where $\omega_n^k = \mathrm{e}^{2\pi\mathrm{i}\frac{k}{n}}$, the integrand has singularities at every n-th root of unity. Then the integral is equal to:

$$\oint_{\partial D(0,2)} \frac{\mathrm{d}z}{\prod_{j=0}^{n-1}\left(z-\omega_j\right)} = \oint_{\partial D(0,2)} \sum_{j=0}^{n-1} \frac{c_j}{z-\omega_j}\,\mathrm{d}z, \qquad (3.1.20)$$

where c_j are the coefficients of the partial fraction decomposition. By the Cauchy–Goursat Formula (Theorem 3.1.8), (3.1.20) becomes:

$$\sum_{k=0}^{n-1} \oint_{\partial D(0,2)} \frac{c_k}{z-\omega_k}\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=0}^{n-1} c_k.$$

Observe that $\sum_{k=0}^{n-1} c_k = \lim_{z\to\infty} \sum_{k=0}^{n-1} \frac{zc_k}{z-\omega_k} = \lim_{z\to\infty} \frac{z}{z^n-1} = 0$ since $n \geq 2$. Therefore,

$$\oint_{\partial D(0,2)} \frac{\mathrm{d}z}{z^n-1} = 0. \qquad \square$$

We have also already seen the utility of parameterization via a polar transformation. Many useful identities in classical calculus can also be derived from concepts in its generalization:

Example 3.1.2: Prove that $\forall n \in \mathbb{N}$,

$$\int_0^{2\pi} \cos^{2n}\theta\,\mathrm{d}\theta = 2\pi \prod_{k=1}^{n} \frac{2k-1}{2k}.$$

Proof: Consider the integral

$$\oint_{\partial\mathbb{D}} \left(z + \frac{1}{z}\right)^{2n} \frac{\mathrm{d}z}{z}.$$

Letting $z = \mathrm{e}^{\mathrm{i}\theta}$, we get $\oint_{\partial\mathbb{D}} \left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right)^{2n} \mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}z = 2^{2n}\mathrm{i}\int_0^{2\pi} \cos^{2n}\theta\,\mathrm{d}\theta$. Alternatively, we can expand the integrand and get

$$\oint_{\partial\mathbb{D}} \sum_{k=0}^{2n} \binom{2n}{k} z^{2k-2n} \frac{\mathrm{d}z}{z} = \sum_{k=0}^{2n} \oint_{\partial\mathbb{D}} \binom{2n}{k} z^{2k-2n-1}\,\mathrm{d}z.$$

When $2k - 2n - 1 \geq 0$, the integrand is holomorphic. The integral is then equal to

$$\binom{2n}{0} \oint_{\partial\mathbb{D}} z^{-2n-1}\,\mathrm{d}z + \binom{2n}{1} \oint_{\partial\mathbb{D}} z^{-2n+1}\,\mathrm{d}z + \dots + \binom{2n}{n} \oint_{\partial\mathbb{D}} \frac{\mathrm{d}z}{z} = 2\pi\mathrm{i}\binom{2n}{n},$$

since all the higher order terms vanish:

$$\oint_{\partial\mathbb{D}} z^{2k-2n-1}\,\mathrm{d}z = \mathrm{i}\int_0^{2\pi} \mathrm{e}^{2\mathrm{i}\theta(k-n)}\,\mathrm{d}\theta = \begin{cases} 0 & \text{if } k < n, \\ 2\pi\mathrm{i} & \text{if } k = n. \end{cases}$$

Therefore,

$$2^{2n}\mathrm{i}\int_0^{2\pi} \cos^{2n}\theta\,\mathrm{d}\theta = 2\pi\mathrm{i}\binom{2n}{n} \iff \int_0^{2\pi} \cos^{2n}\theta\,\mathrm{d}\theta = \frac{2\pi(2n)!}{2^{2n}(n!)^2} = \frac{2\pi\prod_{k=1}^{2n} k}{\prod_{k=1}^{n}(2k)^2}.$$

From simple cancellation, we then have

$$2\pi\prod_{k=1}^{n}\frac{2k-1}{\prod_{k=1}^{n}}(2k) = 2\pi\prod_{k=1}^{n}\frac{2k-1}{2k}. \qquad \square$$

Example 3.1.3 *(Cauchy–Goursat Formula on the Exterior)*: Let $\gamma \subset \mathbb{C}$ be a simple closed curve, and suppose that $f : \operatorname{ext}(\gamma) \to \mathbb{C}$ is holomorphic and continuous on $\overline{\operatorname{ext}(\gamma)} = \mathbb{C} \setminus \operatorname{int}(\gamma)$, where int and ext respectively denote the interior and exterior as in Theorem 1.2.5.

1 If f has a removable singularity at ∞, or if $w = \lim_{z\to\infty} f(z)$ exists and is finite, then $\forall z \in \mathbb{C} \setminus \gamma$,

$$\frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \begin{cases} w & \text{if } z \in \operatorname{int}(\gamma), \\ w - f(z) & \text{if } z \in \operatorname{ext}(\gamma). \end{cases}$$

2 If γ encloses the origin, then $\forall z \in \mathbb{C} \setminus \gamma$,

$$\frac{1}{2\pi\mathrm{i}}\oint_\gamma z\frac{f(\zeta)}{z\zeta - \zeta^2}\,\mathrm{d}\zeta = \begin{cases} 0 & \text{if } z \in \operatorname{int}(\gamma), \\ f(z) & \text{if } z \in \operatorname{ext}(\gamma). \end{cases} \tag{3.1.21}$$

Proof:

1 By the compactness of γ, it can be completely contained within a sufficiently large disk centered at the origin ($\gamma \subset D(0, R)$). Then by applying Theorem 3.1.8 or Theorem 3.1.7 on the set $D(0, R) \cap \operatorname{ext}(\gamma) = D(0, R) \setminus \overline{\operatorname{int}(\gamma)}$, we get that

$$\frac{1}{2\pi\mathrm{i}}\oint_{\partial D(0,R)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \begin{cases} 0 & \text{if } z \in \operatorname{int}(\gamma), \\ f(z) & \text{if } z \in D(0, R) \cap \operatorname{ext}(\gamma). \end{cases}$$

By letting $R \to \infty$ and letting $\zeta = R\mathrm{e}^{\mathrm{i}\theta}$, we get that

$$\frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi}\lim_{R\to\infty}\int_0^{2\pi} \frac{f(R\mathrm{e}^{\mathrm{i}\theta})}{1 - \frac{z}{R\mathrm{e}^{\mathrm{i}\theta}}}\,\mathrm{d}\theta - \begin{cases} 0 & \text{if } z \in \operatorname{int}(\gamma), \\ f(z) & \text{if } z \in \operatorname{ext}(\gamma). \end{cases}$$

By the continuity of f on $\partial D(0, R)$, it attains its maximum M. For sufficiently large R, $\left|1 - \frac{z}{R\mathrm{e}^{\mathrm{i}\theta}}\right|$ attains a positive minimum. Then the integrand is uniformly bounded in R and θ, and hence the order of the limit and the integral may be exchanged. Hence,

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta &= \frac{1}{2\pi}\int_0^{2\pi} \frac{w}{1 - \lim_{R\to\infty}\frac{z}{R\mathrm{e}^{\mathrm{i}\theta}}}\,\mathrm{d}\theta - \begin{cases}0 & \text{if } z \in \operatorname{int}(\gamma),\\ f(z) & \text{if } z \in \operatorname{ext}(\gamma)\end{cases}\\ &= \begin{cases}w & \text{if } z \in \operatorname{int}(\gamma),\\ w - f(z) & \text{if } z \in \operatorname{ext}(\gamma)\end{cases}\end{aligned}$$

as expected.

2 Under the partial fraction decomposition of (3.1.21), we get that

$$\begin{aligned}I &= \oint_\gamma z\frac{f(\zeta)}{z\zeta - \zeta^2}\,\mathrm{d}\zeta = \oint_\gamma \left(\frac{f(\zeta)}{\zeta} - \frac{f(\zeta)}{\zeta - z}\right)\mathrm{d}\zeta\\ &= \int_0^{2\pi}\left(f\left(R\mathrm{e}^{\mathrm{i}\theta}\right) - \frac{f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)}{1 - \frac{z}{R\mathrm{e}^{\mathrm{i}\theta}}}\right)\mathrm{d}\theta + \begin{cases}0 & \text{if } z \in \operatorname{int}(\gamma),\\ 2\pi\mathrm{i}f(z) & \text{if } z \in \operatorname{ext}(\gamma) \cap D(0, R)\end{cases}\end{aligned} \tag{3.1.22}$$

when $\gamma \subset D(0, R)$. We will analyze the first integral as $R \to \infty$. By the triangle and reverse triangle inequalities,

$$\begin{aligned}\left|\int_0^{2\pi}\left(f\left(R\mathrm{e}^{\mathrm{i}\theta}\right) - \frac{f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)}{1 - \frac{z}{R\mathrm{e}^{\mathrm{i}\theta}}}\right)\mathrm{d}\theta\right| &\leq \int_0^{2\pi}\left|\frac{z}{R\mathrm{e}^{\mathrm{i}\theta} - z}\right|\mathrm{d}\theta\\ &\leq \int_0^{2\pi}\frac{|z|}{R - |z|}\,\mathrm{d}\theta = \frac{2\pi|z|}{R - |z|} \to 0.\end{aligned}$$

By substituting the result into (3.1.22), and letting $R \to \infty$, we get that

$$\frac{1}{2\pi\mathrm{i}}\oint_\gamma z\frac{f(\zeta)}{z\zeta - \zeta^2}\,\mathrm{d}\zeta = \begin{cases}0 & \text{if } z \in \operatorname{int}(\gamma),\\ f(z) & \text{if } z \in \operatorname{ext}(\gamma)\end{cases}$$

as desired. ☐

3.2 Analyticity and Holomorphy

The Cauchy–Goursat Formula (Theorem 3.1.8) can also be generalized into a result that equates complex integration and differentiation:

Theorem 3.2.1 *(Cauchy–Goursat)*: Let $U \subset \mathbb{C}$ be an open region bounded by a simple closed boundary ∂U, and let $f : U \to \mathbb{C}$ be holomorphic and continuous over $\overline{U}$. Then $\forall z \in U$, $\forall n \in \mathbb{N}$, $f^{(n)}(z)$ exists, and

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta. \tag{3.2.1}$$

Additionally, since U is open, $\forall a \in U$, $\forall r > 0$ such that the closed disk $\overline{D(a, r)} \subset U$, f has the uniformly and absolutely convergent Taylor expansion

$$f(\zeta) = \sum_{j=0}^{\infty} a_j (z - a)^j, \tag{3.2.2}$$

where

$$a_j = \frac{1}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{(\zeta - z)^{j+1}} \, \mathrm{d}\zeta \tag{3.2.3}$$

for $z \in \overline{D(a, r)}$.

Proof: $\forall a \in U$, $\forall z \in D(a, r) \subset U$, by Theorem 3.1.8,

$$\begin{aligned} f(z) - f(a) &= \frac{1}{2\pi \mathrm{i}} \oint_{\partial U} \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - a} \right) \mathrm{d}\zeta \\ &= \frac{z - a}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta) \, \mathrm{d}\zeta}{(\zeta - z)(\zeta - a)}, \end{aligned}$$

and dividing by $z - a$, the above is equal to

$$\frac{f(z) - f(a)}{z - a} = \frac{1}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta) \, \mathrm{d}\zeta}{(\zeta - z)(\zeta - a)}.$$

Since

$$\begin{aligned} \frac{f(z) - f(a)}{z - a} - \frac{1}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta) \, \mathrm{d}\zeta}{(\zeta - a)^2} &= \frac{1}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{\zeta - a} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - a} \right) \mathrm{d}\zeta \\ &= \frac{z - a}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{(\zeta - z)(\zeta - a)^2} \, \mathrm{d}\zeta, \end{aligned} \tag{3.2.4}$$

Let d be the distance from a to ∂U; then $0 < r < d$. Then since $|z - a| < r$ and $|\zeta - a| \geq d$, $|\zeta - z| \geq d - r$. Then the absolute value of the integrand of (3.2.4) is bounded above by $\frac{M}{d^2(d-r)}$, where M is the maximum of $|f(\zeta)|$, which exists by Theorem 1.2.13. Then,

$$\left| \frac{z - a}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{(\zeta - z)(\zeta - a)^2} \, \mathrm{d}\zeta \right| \leq \frac{|z - a|}{2\pi} \frac{M}{d^2(d - r)} \oint_{\partial U} |\mathrm{d}\zeta|.$$

As $z \to a$, the difference vanishes, and therefore,

$$f'(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{(\zeta - a)^2} \,\mathrm{d}\zeta.$$

Now inductively assume that (3.2.1) is true for a given $n = k \in \mathbb{N}$, or

$$f^{(k)}(z) = \frac{k!}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{(\zeta - z)^{k+1}} \,\mathrm{d}\zeta.$$

Notice the expansion of the kernel, convergent since $|z - z_0| < |\zeta - a|$:

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - a} \cdot \frac{\zeta - a}{\zeta - a + a - z} = \frac{1}{\zeta - a} \cdot \frac{1}{1 - \frac{z-a}{\zeta - a}} \\ &= \frac{1}{\zeta - a} \sum_{j=0}^{\infty} \left(\frac{z - a}{\zeta - a} \right)^j. \end{aligned} \tag{3.2.5}$$

Then,

$$\begin{aligned} f^{(k)}(z) &= \frac{k!}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{(\zeta - z)^{k+1}} \,\mathrm{d}\zeta \\ &= \frac{k!}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{(\zeta - a)^{k+1}} \left(\sum_{j=0}^{\infty} \left(\frac{z - a}{\zeta - a} \right)^j \right)^{k+1} \,\mathrm{d}\zeta \\ &= f^{(k)}(z_0) + \frac{k!(k+1)(z-a)}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{(\zeta - a)^{k+2}} \,\mathrm{d}\zeta \\ &\quad + \mathcal{O}(|z - a|^2), \end{aligned}$$

where the remainder terms $\mathcal{O}(|z - a|^2)$ resemble

$$(z - a)^2 \frac{k!}{2\pi \mathrm{i}} \left[(k+1) + \binom{k+1}{2} \right] \oint_{\partial U} \frac{f(\zeta)}{(\zeta - a)^{k+3}} \,\mathrm{d}\zeta + \mathcal{O}(|z - a|^3).$$

The difference quotient is equal to

$$\frac{f^{(k)}(z) - f^{(k)}(a)}{z - a} = \frac{(k+1)!}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{(\zeta - a)^{k+2}} \,\mathrm{d}\zeta + \mathcal{O}(|z - a|).$$

As $z \to a$, the remainder terms vanish, and

$$f^{(k+1)}(a) = \frac{(k+1)!}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{(\zeta - a)^{k+2}} \,\mathrm{d}\zeta.$$

By induction, (3.2.1) is valid. By substituting (3.2.5) into Theorem 3.1.8, we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi \mathrm{i}} \oint_{\partial U} \frac{f(\zeta)}{\zeta - a} \sum_{j=0}^{\infty} \left(\frac{z-a}{\zeta - a} \right)^j \mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\partial U} \sum_{j=0}^{\infty} (z-a)^j \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - a)^{j+1}}. \end{aligned}$$

Because $f(\zeta)$ is continuous over ∂U, it is bounded by a constant M. Additionally, since $|z - a| < r < |\zeta - a|$ and consequently $c = \left| \frac{z-a}{\zeta - a} \right| < 1$, the sum is termwise uniformly bounded by the convergent series

$$\sum_{j=0}^{\infty} M \frac{c^j}{r}.$$

By the Weierstrass M–Test (Theorem 2.3.2), the integrand uniformly converges, and we can justify

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \oint_{\partial U} \sum_{j=0}^{\infty} (z - z_0)^j \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \,\mathrm{d}\zeta &= \frac{1}{2\pi \mathrm{i}} \sum_{j=0}^{\infty} \oint_{\partial U} (z - z_0)^j \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \,\mathrm{d}\zeta \\ &= \sum_{j=0}^{\infty} a_j (z - z_0)^j, \end{aligned}$$

which verifies (3.2.2) and (3.2.3). ☐

Remark: By induction, we have shown that assuming the existence of the first order derivative of a holomorphic function f, the n-th order derivative of f exists $\forall n \in \mathbb{N}$ and is holomorphic over the same region as $f^{(n-1)}$. Furthermore, if f is holomorphic, then $\forall z \in U$, there exists an open disk enclosing z such that f has a convergent Taylor series expansion. This property is known as *analyticity*, and Theorem 3.2.1 tells us that all holomorphic functions are analytic. Analytic functions can be expanded into power series, which are termwise differentiable, and therefore complex differentiable. Thus, analyticity and holomorphy are logically equivalent, which is a fundamental difference between real and complex functions.

The differentiation formula above can be thought of as a generalization of Theorem 3.1.8, and provides similar utility in the evaluation of integrals:

Example 3.2.1: A *Legendre polynomial* is a polynomial whose explicit equation is given by

$$P_n(z) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} (z^2 - 1)^n. \tag{3.2.6}$$

Prove the integral form

$$P_n(z) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{(\zeta^2 - 1)^n}{2^n (\zeta - z)^{n+1}} \, \mathrm{d}\zeta,$$

where γ is a simple closed curve enclosing z.

Proof: By applying Cauchy–Goursat (Theorem 3.2.1) on (3.2.6), we get that

$$P_n(z) = \frac{1}{2^{n+1} \pi \mathrm{i}} \oint_\gamma \frac{(\zeta^2 - 1)^n}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta,$$

as desired. □

Theorem 3.2.2 *(Cauchy's Estimate)*: For a function $f : U \to \mathbb{C}$ holomorphic over $U \subseteq \mathbb{C}$ and $\forall z_0 \in U$ and $\forall R > 0$ such that $\overline{D(z_0, R)} \subseteq U$, $\forall n \in \mathbb{N}$,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n! M}{R^n},$$

where

$$M = \max_{z \in \overline{D(z_0, R)}} |f(z)|.$$

Proof: By the Differentiation Formula (Theorem 3.2.1), $\forall n \in \mathbb{N}$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_{\partial D(z_0, R)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta.$$

Because $f(z)$ is continuous over the boundary $\partial D(z_0, R)$, it is bounded by M. Thus,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{M}{(\mathrm{e}^{\mathrm{i}\theta} R)^{n+1}} \mathrm{e}^{\mathrm{i}\theta} R \, \mathrm{d}\theta = \frac{n! M}{R^n},$$

as desired. □

Theorem 3.2.5 will profoundly generalize this statement significantly. The relationship between the derivatives of a holomorphic function and the function itself is an important property of holomorphic functions.

Example 3.2.2: Let f be entire and $\forall z \in \mathbb{C}$, $|f(z)| \leq M \mathrm{e}^{|z|}$. Prove that $\forall n \in \mathbb{N}$, $|f(0)| \leq M$ and

$$\left|f^{(n)}(0)\right| \leq M n! \left(\frac{\mathrm{e}}{n}\right)^n.$$

Proof: $|f(0)| \leq M$ is obviously true by letting $z = 0$. Then $\forall R > 0$, by Cauchy's Estimate (Theorem 3.2.2),

$$\left|f^{(n)}(0)\right| \leq Mn!\frac{\mathrm{e}^R}{R^n}.$$

By letting $R = n$, the conclusion follows. In fact, this is the tightest possible inequality. Consider $\varphi(R) = Mn!\frac{\mathrm{e}^R}{R^n}$ to be a function of R. It attains its minimum as its derivative vanishes:

$$\varphi'(R) = Mn!\frac{\mathrm{e}^R R^n - n\mathrm{e}^R R^{n-1}}{R^{2n}} = 0 \Longleftrightarrow R^n = nR^{n-1} \Longleftrightarrow R = n.$$

To confirm it as a minimum, we calculate the second order derivative:

$$\varphi''(R) = Mn!\mathrm{e}^R\left(\frac{1}{R^n} - \frac{2n}{R^{n+1}} + \frac{n(n+1)}{R^{n+2}}\right) \Longrightarrow \varphi''(n) = M(n-1)!\frac{\mathrm{e}^n}{n^n},$$

which is positive and convex. □

The following theorem, albeit originally proven by Cauchy in 1844, shows a fundamental difference between holomorphic functions on proper subsets of $\mathbb{C}$ and entire functions.

Theorem 3.2.3 *(Liouville)*: Any bounded entire function is constant.

Proof: Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Then, $\forall z_0 \in \mathbb{C}$, $\forall R > 0$, f is holomorphic over $\overline{D(z_0, R)}$. By Theorem 3.2.2,

$$|f'(z_0)| \leq \frac{M}{R},$$

where $M = \sup_{z\in\mathbb{C}}|f(z)|$. By letting $R \to \infty$, $f'(z_0)$ where z_0 is any arbitrary value in $\mathbb{C}$. Therefore, $f(z)$ is constant. □

Proof Alternative Proof: Let $a, b \in \mathbb{C}$ be distinct and arbitrarily chosen. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and bounded such that $|f| \leq M$ for some $M > 0$. Let $R > |a|, |b|$. Since $a \neq b$, $\exists\varepsilon > 0$ such that $\overline{D(a,\varepsilon)} \cup \overline{D(b,\varepsilon)} = \varnothing$. By the Cauchy–Goursat Theorem (Theorem 3.1.7), we have

$$\oint_{\partial D(0,R)} \frac{f(z)}{(z-a)(z-b)}\,\mathrm{d}z = \left(\oint_{\partial D(a,\varepsilon)} + \oint_{\partial D(b,\varepsilon)}\right)\frac{f(z)}{(z-a)(z-b)}\,\mathrm{d}z.$$

Since $z \mapsto \frac{f(z)}{z-a}$ is holomorphic on the disk centered at b and $z \mapsto \frac{f(z)}{z-b}$ is holomorphic on the disk centered at a, by the Cauchy–Goursat Formula (Theorem 3.1.8), we have

$$\oint_{\partial D(0,R)} \frac{f(z)}{(z-a)(z-b)}\,\mathrm{d}z = 2\pi\mathrm{i}\left(\frac{f(b)}{b-a} + \frac{f(a)}{a-b}\right).$$

On the contrary, we also have

$$\begin{aligned}\left|\left(\oint_{\partial D(a,\varepsilon)}+\oint_{\partial D(b,\varepsilon)}\right)\frac{f(z)}{(z-a)(z-b)}\,\mathrm{d}z\right| &\leq M\oint_{\partial D(0,R)}\frac{|\mathrm{d}z|}{|z-a||z-b|}\\ &= \frac{2\pi MR}{(R-a)(R-b)}\\ &\to 0 \quad \text{as} \quad R\to\infty.\end{aligned}$$

We conclude that

$$\frac{2\pi\mathrm{i}}{b-a}(f(b)-f(a))=0$$

for all distinct complex a and b. Hence, f is a constant function. □

Theorem 3.2.4 *(Morera)*: Let $U \subseteq \mathbb{C}$ and $f : U \to \mathbb{C}$ be continuous over U. If for any closed triangular contour $\gamma \subset U$,

$$\oint_\gamma f(\zeta)\,\mathrm{d}\zeta = 0,$$

then f is holomorphic over U.

Proof: Let $a \in U$ be arbitrary. Since U is open, $\exists r > 0$ such that $\overline{D} = \overline{D(a,r)} \subset U$. Define

$$F(z)=\int_a^z f(\zeta)\,\mathrm{d}\zeta,$$

where the path is a straight line segment, and F is well-defined for $z \in D$. Now

$$\begin{aligned}F'(z) &= \lim_{\Delta z\to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z}\\ &= \lim_{\Delta z\to 0}\frac{\left[\int_a^{z+\Delta z}+\int_z^a+\left(\int_{z+\Delta z}^z+\int_z^{z+\Delta z}\right)\right]f(\zeta)\,\mathrm{d}\zeta}{\Delta z}.\end{aligned}$$

Note that the first three integrals sum to form a closed triangular curve and hence vanish by assumption. Therefore,

$$F'(z)=\lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_z^{z+\Delta z}f(\zeta)\,\mathrm{d}\zeta.$$

By the continuity of f at z, for any $\varepsilon > 0$, $\exists\delta > 0$ such that $|\zeta - z| < \delta \implies |f(\zeta) - f(z)| < \varepsilon$. Then, for $|\Delta z| < \delta$,

$$\left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta)\,\mathrm{d}\zeta - f(z)\right| = \left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z} (f(\zeta)-f(z))\,\mathrm{d}\zeta\right| \leq \varepsilon.$$

Thus, $F'(z) = f(z)$ for all $z \in D$. Since a was arbitrary, f is holomorphic over U. □

Theorem 3.2.5: Let $U \subseteq \mathbb{C}$ be open, let $K \subset U$ be compact and $V \supset K$ be open such that $\overline{V} \subset U$ is compact ($V \supseteq K$ is relatively compact in U). Let $f(z)$ be holomorphic in U. Then there exists a sequence $\{c_n\} \subset \mathbb{R}$ dependent only on K and V (independent of f and z) such that $\forall n \in \mathbb{N}$,

$$\sup_{z\in K}\left|f^{(n)}(z)\right| \leq c_n \|f\|_{L^1(V)}, \tag{3.2.7}$$

where $\|f\|_{L^p(V)}$ denotes

$$\left(\int_V |f(z)|^p \,\mathrm{d}x \wedge \mathrm{d}y\right)^{\frac{1}{p}}.$$

Proof: Let $\varphi \in C^\infty(\mathbb{C})$ satisfy $\operatorname{supp}(\varphi) \subset V$ and be identically equal to 1 over some open neighborhood W of K relatively compact in V. Since $f \in C^\infty(U)$, by the Cauchy–Pompeiu Theorem (Theorem 3.1.3) on $f(z)\varphi(z) \in C^\infty\left(\overline{U}\right)$,

$$f(z)\varphi(z) = \frac{1}{2\pi\mathrm{i}}\left(\oint_{\partial U} \frac{f(\zeta)\varphi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \int_U \frac{\partial f(\zeta)\varphi(\zeta)}{\partial\overline{\zeta}} \cdot \frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{\zeta - z}\right).$$

By the product rule,

$$\frac{\partial f(\zeta)\varphi(\zeta)}{\partial\overline{\zeta}} = \frac{\partial\varphi(\zeta)}{\partial\overline{\zeta}} f(\zeta),$$

and since $\partial U \subset \mathbb{C} \setminus \operatorname{supp}(\varphi)$, the first term vanishes, resulting in

$$f(z)\varphi(z) = -\frac{1}{2\pi\mathrm{i}}\int_U \frac{\partial\varphi(\zeta)}{\partial\overline{\zeta}} f(\zeta) \cdot \frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{\zeta - z}.$$

Let K_1 denote $\operatorname{supp}\left(\frac{\partial\varphi(\zeta)}{\partial\overline{\zeta}}\right)$, and $\forall z \in K$, $\varphi(z) = 1$. Therefore,

$$f(z) = \frac{1}{2\pi\mathrm{i}}\int_{K_1} f(\zeta) \cdot \frac{\partial\varphi(\zeta)}{\partial\overline{\zeta}} \cdot \frac{\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}}{\zeta - z}.$$

We can differentiate within the integral as $f(\zeta) \cdot \frac{\partial\varphi(\zeta)}{\partial\overline{\zeta}}$ is C^∞ and bounded over K_1, and thus the integrand is uniformly bounded by an integrable function independent of ζ:

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \int_{K_1} f(\zeta) \cdot \frac{\partial \varphi(\zeta)}{\partial \overline{\zeta}} \cdot \frac{\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}}{(\zeta - z)^{n+1}},$$

and by the triangle inequality,

$$\left|f^{(n)}(z)\right| \leq \frac{n!}{2\pi} \int_{K_1} |f(\zeta)| \left|\frac{\partial \varphi(\zeta)}{\partial \overline{\zeta}}\right| \frac{\left|\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}\right|}{|\zeta - z|^{n+1}}.$$

Notice that over $W, \varphi = 1, \varphi' = 0$, and is disjoint from K_1 (or that $W \cap K_1 = \varnothing$). Then, the distance between W and K is positive and the two are disjoint. Therefore, $\exists M > 0$ such that

$$\frac{1}{|\zeta - z|} \leq M,$$

and thus,

$$\left|\frac{\partial \varphi(\zeta)}{\partial \overline{\zeta}}\right| \frac{1}{|\zeta - z|^{n+1}}$$

can be bounded by a sequence $\{c_n'\}$, independent of f and dependent only on n and the sets K and V. Then,

$$\left|f^{(n)}(z)\right| \leq \frac{n!}{2\pi} \int_{K_1} c_n' |f(\zeta)| \left|\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}\right| = \frac{n!}{\pi} \int_{K_1} c_n' |f(\zeta)| |\mathrm{d}x \wedge \mathrm{d}y|.$$

Because K_1 is compact, it has a finite area $\operatorname{area}(K_1)$, and we can define a new sequence $c_n = n! c_n' \operatorname{area}(K_1)/\pi$ to find that

$$\left|f^{(n)}(z)\right| \leq c_n \int_{K_1} |f(\zeta)| |\mathrm{d}x \wedge \mathrm{d}y| \leq c_n \int_{V} |f(\zeta)| |\mathrm{d}x \wedge \mathrm{d}y|.$$

The problem now stands to prove that $\varphi(z)$ exists in the first place, which requires a topological argument to be later discussed in Theorem 3.2.1.6. □

Corollary 3.2.5.1: Let $U \subseteq \mathbb{C}$ be open, let $K \subset U$ be compact and $V \supset K$ be open such that $\overline{V} \subset U$. For any holomorphic function $f(z)$ in U, there exist constants (independent of z and f) $\{c_n\}$ such that

$$\sup_{z \in K} \left|f^{(n)}(z)\right| \leq c_n \sup_{z \in V} |f(z)|.$$

Proof: Starting from (3.2.7), observe that

$$c_n \|f\|_{L^1(V)} \leq c_n \operatorname{area}(V) \sup_{z \in V} |f(z)|,$$

and we can define a new set of constants equal to $c_n \operatorname{area}(V)$, which are still independent of z. $\square$

For the next theorem we will briefly introduce the concept of *analytic continuation.*

Definition 3.2.1 *(Analytic Continuation)*: Let $U \subseteq \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$ be holomorphic. Let $V \subseteq \mathbb{C}$ be open with $U \subseteq V$. A function

$$F : V \to \mathbb{C}$$

is an *analytic continuation* of f to V if:
1 F is holomorphic on V, and
2 $F \equiv f$ on U.

The concept of analytic continuation and its consequent problems and properties will be discussed in detail in a later chapter. For now, we will prove a theorem that is a direct consequence of the Cauchy–Goursat Differentiation Formula (Theorem 3.2.1) and the existence of holomorphic functions with removable singularities.

Theorem 3.2.6 *(Riemann)*: Let $D^*(z_0, r) = D(z_0, r) \setminus \{z_0\}$ (known as a punctured disk), and $f : D^*(z_0, r) \to \mathbb{C}$ be holomorphic and bounded. Then f can be analytically continued to $D(z_0, r)$.

Proof: Define the auxiliary function

$$\varphi(z) = \begin{cases} (z - z_0)^2 f(z) & \text{if } z \in D^*(z_0, r) \\ 0 & \text{if } z = z_0. \end{cases}$$

$\varphi(z)$ is bounded and continuously differentiable on $D(z_0, r)$ and satisfies the Cauchy–Riemann Equations since

$$\lim_{z \to z_0} \frac{\varphi(z) - \varphi(z_0)}{z - z_0} = \frac{(z - z_0)^2 f(z)}{z - z_0} = \lim_{z \to z_0} (z - z_0) f(z) = 0,$$

meaning that $\frac{\mathrm{d}\varphi}{\mathrm{d}z}(z_0) = 0$. For $z \in D^*(z_0, r)$,

$$\varphi'(z) = 2(z - z_0) f(z) + (z - z_0)^2 f'(z).$$

As $z \to z_0$, $\varphi(z) \to 0$, meaning that φ is holomorphic over $D(z_0, r)$. By Theorem 3.2.1,

$$\varphi(z) = \sum_{j=2}^{\infty} a_j (z - z_0)^j,$$

which is convergent over $D(z_0, r)$. Then we can define

$$\tilde{f}(z) = \frac{\varphi(z)}{(z - z_0)^2} = \sum_{j=0}^{\infty} a_{j+2}(z - z_0)^j$$

over the same disk of convergence. Over the punctured disk, $\tilde{f}(z) = f(z)$, and therefore $\tilde{f}$ is an analytic continuation of f. ☐

3.2.1 Topology, Partitions of Unity, and the Existence of Bump Functions

Definition 3.2.1.1 *(Topological Space)*: A *topological space* is a pair (X, τ), where X is a set and τ is a collection of subsets of X satisfying the following properties:

1 $\varnothing \in \tau$ and $X \in \tau$.
2 The union of any (possibly infinite) collection of sets in τ is also in τ.
3 The intersection of any finite collection of sets in τ is also in τ.

The collection τ is called a *topology* on X, and its elements are referred to as *open sets* under the topology τ.

Obviously the statement "let X be a topological space" itself has little meaning. However, when the topology is implicitly obvious or the space is describable without it, then it may be verbally elided.

The implied topology of a subspace A of (X, τ) is given by the intersection of each set in τ with A.

Definition 3.2.1.2: A subset A of a topological space X is *closed* iff $X \setminus A$ is open.

It is immediate from definition that the trivial sets X and $\varnothing$ are always closed. It is equally trivial from definition that the union of finitely many closed sets is closed, and the intersection of any collection of closed sets is closed.

If $\exists U \in \tau$ such that $x \in U$, then U is an (open) *neighborhood* of x. If $\forall x, y \in X$ (such that $x \neq y$) have disjoint neighborhoods, then X is a *Hausdorff space*.

The following discussions involved with topological spaces here will always be of Hausdorff spaces, although making such distinction is important for future extensibility.

Definition 3.2.1.3: A topological space X is *compact* iff every open cover has a finite subcover. For a topological space X, a set $A \subseteq X$ is *compact* iff every open cover has a finite subcover.

Proposition 3.2.1.1: Suppose X is a Hausdorff topological space and let $A \subseteq X$ be compact. Then A is closed in X.

Proof: Let $x \in X \setminus A$ be fixed. For each $a \in A$, since X is Hausdorff, there exist disjoint neighborhoods U_a and V_a with $x \in U_a$ and $a \in V_a$. The set

$$\bigcup_{a \in A} V_a \supseteq A$$

covers A, which by assumption, has a finite subcover

$$\bigcup_{k=1}^{n} V_{a_k} \supseteq A, \quad \forall k \in \mathbb{N}_{\leq n}, a_k \in A.$$

Moreover, the intersection

$$U_x = \bigcap_{k=1}^{n} U_{a_k}$$

is an open neighborhood of x and by construction, it is disjoint from the finite subcover. Since it is disjoint from a superset of A, it lies entirely in $X \setminus A$.

For each $x \in X \setminus A$, construct open U_x accordingly. Then we obtain

$$X \setminus A \subseteq \bigcup_{x \in X \setminus A} U_x \subseteq X \setminus A,$$

where the sandwiched union is open. Therefore, A is closed. □

Proposition 3.2.1.2: If X is a compact space and $A \subseteq X$ is closed, then A is compact.

Proof: Let $\mathcal{U}$ be an open cover of A in X. Since $X \setminus A$ is open, the set

$$\{U \cup (X \setminus A) : U \in \mathcal{U}\}$$

openly covers X. Then a finite subcover

$$\{U_k \cup (X \setminus A)\}_{k \in \mathbb{N}_{\leq n}}$$

exists and covers X. The refinement $\{U_k\}_{k \in \mathbb{N}_{\leq n}}$ then covers A. □

Definition 3.2.1.4: A point a is an accumulation point of a set A in a topological space X iff any open U with $a \in U$ implies that $U \cap A$ contains a point other than a.

Proposition 3.2.1.3: A set A in a topological space X is closed iff it contains all its accumulation points.

Proof: We first prove the forward implications under the assumption that A is closed. Since $X \setminus A$ is open, and suppose for contradiction, that $a \in X \setminus$

A. Then for $a \in U = X \setminus A$ open, $U \cap A = \varnothing$ (and hence a cannot be an accumulation point by contradiction of definition).

Assume the converse assumption that A contains all its accumulation points. Let $x \in X \setminus A$ be arbitrary. By assumption, x is not an accumulation point of A. Hence, for some open set $U \supseteq \{x\}$, $U \cap A$ does not contain a point other than x (which it also cannot contain), implying that $U \cap A = \varnothing$, and hence $U \subseteq X \setminus A$.

For each $x \in X \setminus A$, we hence construct some open neighborhood fully contained in $X \setminus A$. Together, they must union (by the definition of a topology) to an open set, being $X \setminus A$. Therefore, A is closed. □

A topology allows the definition and general conceptualization of continuity, convergence, and connectivity in a general setting, without necessarily relying on a notion of distance (a metric).

Definition 3.2.1.5: A function $f : X \to Y$ between two topological spaces is said to be *continuous* if the *pre-image* of every open set in Y,

$$\{x \in X : f(x) \in Y\},$$

is an open set in X.

For the case of metric spaces, this generalizes the epsilon–delta notion of continuity.

Example 3.2.1.1: Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ 0 \text{ if } x < 0. \end{cases}$$

We equip both the domain and codomain with the standard topology on $\mathbb{R}$. Let $V = (.5, 1.5) \subseteq \mathbb{R}$. Then the pre-image of V is

$$f^{-1}(V) = \{x \in \mathbb{R} : f(x) \in V\} = \mathbb{R}_{\geq 0},$$

which is not an open set in the standard topology on $\mathbb{R}$. Thus, f is not continuous.

For two topological spaces X and Y, a function $f : X \to Y$ is a *homeomorphism* (also known as a *bicontinuous function*) if it is a bijection such that both f and f^{-1} are continuous. If such a function exists, then X and Y are *homeomorphic*.

The function $f : [0, 2\pi) \to S^1$ with $f(t) = (\cos(t), \sin(t))$ is indeed continuous, but the inverse $f^{-1}(x_1, x_2)$ is discontinuous at $(x_1, x_2) = (1, 0)$.

Proposition 3.2.1.4: Let (X, τ_1), (Y, τ_2) be two topological spaces. Then for $f : X \to Y$, the following conditions are equivalent:

1 f is continuous.
2 If $A \subseteq Y$ is closed, then the pre-image $f^{-1}(A)$ is closed.
3 If $a \in X$ and $A \in \tau_2$ is an open neighborhood of $f(a)$ in Y, then there is some $U \in \tau_1$ that is a neighborhood of a such that $f(U) \subseteq A$.

Proof: We first show that Part 1 implies Part 2. By continuity, for $A \subseteq Y$ closed, $Y \setminus A \in \tau_2$, then

$$f^{-1}(Y \setminus A) = \{x \in X : f(x) \in Y \setminus A\} = X \setminus f^{-1}(A).$$

Assume the conditions of Part 2 for the converse. Let $U \in \tau_2$ be open, $Y \setminus U$ closed, then $f^{-1}(Y \setminus U)$ is closed. Similar logic shows

$$f^{-1}(Y \setminus U) = X \setminus f^{-1}(U),$$

which implies $f^{-1}(U)$ is open.

Next we aim to show that continuity implies Part 3. By assumption, the pre-image of any open $A \subseteq Y$ is $f^{-1}(A)$ and open and $a \in f^{-1}(A)$. The property is complete under $U = f^{-1}(A)$.

Assume the conditions of Part 3 for the converse. Let $A \in \tau_2$ be arbitrary. We aim to show that $f^{-1}(A) \in \tau_1$. If $f^{-1}(A) = \varnothing$, then the conclusion is satisfied trivially. Hence, assume that $\exists a \in f^{-1}(A)$. For any such a, there exists a neighborhood $U_a \in \tau_1$ such that $f(U_a) \subseteq A$. Hence $U_a \subseteq f^{-1}(A)$ for any $a \in f^{-1}(A)$. Therefore, we obtain

$$\bigcup_{a \in f^{-1}(A)} U_a \subseteq f^{-1}(A),$$

$$\bigcup_{a \in f^{-1}(A)} U_a \supseteq \bigcup_{a \in f^{-1}(A)} \{a\} = f^{-1}(A) \Longrightarrow \bigcup_{a \in f^{-1}(A)} U_a = f^{-1}(A).$$

By the definition of topologies,

$$\bigcup_{a \in f^{-1}(A)} U_a \in \tau_1.$$

□

Definition 3.2.1.6 *(Basis for a Topology)*: Let X be a set. A *basis* for a topology on X is a collection $\mathfrak{B}$ of subsets of X satisfying

1 $\bigcup_{B \in \mathfrak{B}} B = X$.
2 For any $B_1, B_2 \in \mathfrak{B}$ and any point $x \in B_1 \cap B_2$, there exists a set $B_3 \in \mathfrak{B}$ such that

$$x \in B_3 \subseteq B_1 \cap B_2.$$

The topology generated by $\mathfrak{B}$ is the collection of all unions of elements of $\mathfrak{B}$.

Definition 3.2.1.7: A *metric space* is a pair (X, d), where X is a set and d is a function from $X \times X$ to $\mathbb{R}_{\geq 0}$, called a *metric*, such that for all $x, y, z \in X$ the following properties hold:
1 $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$ (positivity).
2 $d(x, y) = d(y, x)$ (symmetry).
3 $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

This in turn implies the reverse triangle inequality:

$$d(x, z) \leq d(x, y) + d(y, z) \Longrightarrow d(x, y) \geq d(x, z) - d(y, z),$$

and similarly,

$$d(y, z) \leq d(x, y) + d(x, z) \Longrightarrow d(x, y) \geq d(y, z) - d(x, z).$$

Definition 3.2.1.8: Let (X, d) be a metric space. The *metric topology induced by d* is the topology τ_d generated by the basis

$$\{B(x, r) : x \in X, r > 0\}$$

comprising the balls

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

The pair (X, τ_d) is the *topological space induced by the metric d*.

We now justify a claim whose triviality we have taken for granted.

Proposition 3.2.1.5: Let (X, d) be a metric space under the induced metric topology. Then for any open set $U \subseteq X$, any point $x \in U$, there exists a ball $B(x, \delta)$ ($\delta > 0$) in U.

Proof: By definition, U lies in the topology for X and is the union of (possibly infinitely many) balls. There then exists some ball $B(x_0, \delta')$ in U that contains x. Let

$$\delta = \delta' - d(x_0, x).$$

Since $d(x_0, x) < \delta'$, for any $y \in B(x, \delta)$, we have

$$d(x_0, y) \leq d(x_0, x) + d(x, y) \leq \delta'.$$

Hence, the open ball $B(x, \delta)$ centered at x lies within $B(x_0, \delta')$. □

Theorem 3.2.1.1: Let $(X, d_x), (Y, d_y)$ be two metric spaces under the metric topology. Then a function $f : X \to Y$ is topologically continuous iff it is epsilon–delta continuous.

Proof: We first imply that topological continuity implies epsilon–delta continuity. For any $x \in X$, $\forall \varepsilon > 0$, the ball $B(f(x), \varepsilon)$ is an open set (it is in the basis) in Y. By Part 3 of Proposition 3.2.1.4, there is some open neighborhood U of x in X such that $f(U) \subset B(f(x), \varepsilon)$. By the previous proposition, there is a ball $B(x, \delta) \subseteq U$. This is equivalent to

$$\varepsilon > 0, x \in X \Longrightarrow \exists \delta = \delta_x > 0 : y \in B(x, \delta) \Longrightarrow f(y) \in B(f(x), \varepsilon).$$

Conversely, assume f is ε–δ continuous. Let $V \subseteq Y$ be open and $x \in f^{-1}(V)$. Since V is open in the metric topology, there exists $\varepsilon > 0$ such that

$$B(f(x), \varepsilon) \subseteq V.$$

By epsilon–delta continuity, there exists $\delta > 0$ such that

$$d_x(x, y) < \delta \Longrightarrow d_y(f(x), f(y)) < \varepsilon,$$

or that

$$y \in B(x, \delta) \Longrightarrow f(y) \in B(f(x), \varepsilon) \subseteq V.$$

Thus, the ball $B(x, \delta_x)$ is an open neighborhood of x in X such that

$$B(x, \delta_x) \subseteq f^{-1}(V) \Longrightarrow f^{-1}(V) \subseteq \bigcup_{x \in f^{-1}(V)} B(x, \delta_x) \subseteq f^{-1}(V).$$

Since the union of open sets is open, the pre-image of any open set is open, and hence f is topologically continuous. □

Theorem 3.2.1.2: Let X be a compact topological space and let Y be a Hausdorff space. If $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

Proof: If $A \subseteq X$ is compact, then the pre-images of any open cover $\mathcal{U}$ of $f(A)$ cover A. Hence, there is a finite subcover

$$\{f^{-1}(U_k) : U_k \in \mathcal{U}, k \in \mathbb{N}_{\leq n}\}$$

covering A. Then

$$\{U_k : U_k \in \mathcal{U}, k \in \mathbb{N}_{\leq n}\}$$

covers $f(A)$, and hence $f(A)$ is compact.

For any closed $C \subseteq X$, Proposition 3.2.1.2 implies C is compact. Hence, $f(C)$ is compact. By Proposition 3.2.1.1, $f(C)$ is closed. Hence, f maps closed sets to closed sets, and the pre-image of any closed set is closed under f^{-1}. Hence, Proposition 3.2.1.4 implies f^{-1} is continuous, thus f is a homeomorphism. □

It is worth noting some motivating examples for which the conclusion fails when certain hypotheses are not satisfied.

Example 3.2.1.2: Let $I = [0, 2\pi)$ be a topological space with a metric $|\cdot|$ under the standard topology (the subspace topology induced by the basis formed with open "balls" or symmetric intervals around each point). Equip the unit circle

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

generated by the metric defined by arc-length (d_{S^1}). Then the continuous bijection $f : I \to S^1$ defined by $f(t) = \mathrm{e}^{\mathrm{i}t}$ is not a homeomorphism.

Proof: The non-continuity of $f^{-1} : S^1 \to I$ is easy to visually see, both topologically and by epsilon–delta. Topologically, select the *relatively* open interval $[0, \pi)$ in I. The pre-image of this set under f^{-1} is $\mathrm{e}^{\mathrm{i}[0,\pi)}$, which is clearly not an open set. This proves that f^{-1} is not continuous (by definition).

For continuity to hold by epsilon–delta, any ε would yield the existence of some δ such that $\forall a, b \in S^1$ with $d_{S^1}(a, b) < \delta$, $\left|f^{-1}(a) - f^{-1}(b)\right| < \varepsilon$.

Let $\varepsilon = \frac{\pi}{2}$. For any $0 < \delta < 2\pi$, the points

$$a = \mathrm{e}^{-\mathrm{i}\frac{\delta}{4}}, b = \mathrm{e}^{\mathrm{i}\frac{\delta}{4}}$$

satisfy

$$d_{S^1}(a, b) = \frac{\delta}{2} < \delta.$$

However,

$$f^{-1}(a) = 2\pi - \frac{\delta}{4}, \quad f^{-1}(b) = \frac{\delta}{4},$$

and

$$\left|f^{-1}(a) - f^{-1}(b)\right| = 2\pi - \frac{\delta}{2} > \pi > \varepsilon.$$

This contradicts the previous statement. □

We now provide a formal definition of the connectivity of sets:

Definition 3.2.1.9: A topological space X is *disconnected* if it can be written as the union of two nonempty disjoint open sets. Otherwise, it is *connected.*

In a topological space X, a subset can be open, closed (the complement of some open set), both (clopen), or neither. The only clopen sets that exist in any topological space X are $\varnothing$ and X iff X is connected. A technique pertinent to many future proofs relies on the following fact:

Theorem 3.2.1.3 *(Connectivity Argument)*: A topological space X is *connected* if and only if X and $\varnothing$ are the only clopen subsets of X.

Proof: Suppose X is connected and let $A \subseteq X$ be clopen. Then A and $X \setminus A$ are both open in X, disjoint, and their union is X. Thus, either $A = \varnothing$ or $X \setminus A = \varnothing$ (i.e. $A = X$).

Conversely, suppose X is disconnected. Then there exist nonempty open sets $U, V \subseteq X$ such that $U \cap V = \varnothing$ and $U \cup V = X$. Thus,

$$U = X \setminus V$$

and

$$V = X \setminus U$$

are both clopen, contradicting the assumption that X and $\varnothing$ are the only clopen subsets. Hence, X must be connected. □

Example 3.2.1.3: The topological space $\mathbb{R}$ under the standard topology has only two clopen sets: $\mathbb{R}$ and $\varnothing$.

Now consider

$$X = \bigcup_{n \in 2\mathbb{Z}} (n, n+1),$$

equipped with the topology τ generated by the basis

$$\{(n, n+1) : n \in 2\mathbb{Z}\}.$$

This space is disconnected. For instance, $(0, 1) \subset X$ is open (as it is in τ) and closed (since its complement in X is

$$\bigcup_{n \in (2\mathbb{Z} \setminus \{0\})} (n, n+1) \in \tau).$$

In fact, every set in τ is clopen.

Proposition 3.2.1.6: The interval $[0, 1]$ (under the subspace topology induced by $\mathbb{R}$) is connected.

Proof: Assume, for contradiction, that there exist two disjoint nonempty open sets $U, V \subset [0, 1]$ such that

$$U \cup V = [0, 1].$$

Without loss of generality, assume $0 \in U$ (otherwise switch U and V). Let

$$a = \inf V.$$

Since U, V are also closed in $[0, 1]$, either $a \in V$ or a is an accumulation point. Either way, a is contained in V by Proposition 3.2.1.3. Assume that $a \neq 0$. Then by openness, there exists some $0 < \delta < a$ such that

$$(a - \delta, a) \subseteq (a - \delta, a + \delta) \cap [0, 1] \subseteq V.$$

In particular,

$$a - \frac{\delta}{2} \in V,$$

which contradicts a being a lower bound of V.

Therefore, $a = 0$. However, since $[0, \delta)$ lies in U for some $\delta > 0$, $a \geq \delta > 0$. Thus, we arrive at a contradiction, and thus V is the empty set. This then shows that $[0, 1]$ is connected. □

Connectivity intuitively means that a space cannot be split into two disjoint open subsets, but is not meaningful in terms of how points within the space relate to each other. In many geometric situations, the notion of *path-connectivity* requires that any two points be joined by a continuous path. We will see that this more concrete condition forces the space to be topologically connected.

Definition 3.2.1.10: A topological space X is said to be *path-connected* iff for any two points $a, b \in X$, there is a continuous function $f : [0, 1] \to X$, where $[0, 1]$ is equipped with the metric topology and $f(0) = a, f(1) = b$.

Theorem 3.2.1.4: A path-connected topological space X is connected.

Proof: Assume path-connectivity and suppose X is disconnected. Then two open nonempty disjoint components $U, V \subset X$ can be found. Let $u \in U, v \in V$ be two arbitrary points. Then there exists $f : [0, 1] \to X$ such that $f(0) = u$, $f(1) = v$. By continuity, the pre-images of U and V, namely $f^{-1}(U)$ and $f^{-1}(V)$ respectively, are disjoint open subsets of $[0, 1]$. Moreover, the pre-image are nonempty as they contain 0 and 1 respectively. This contradicts the connectivity of $[0, 1]$ in Proposition 3.2.1.6. □

Definition 3.2.1.11 *(Exhaustion by Compact Sets)*: For a topological space X, an *exhaustion by compact sets* is a nested sequence of compact sets $\{K_n\}_{n \in \mathbb{N}} \subseteq X$ such that $K_n \subset \overset{\circ}{K}_{n+1}$ for all $n \in \mathbb{N}$ and

$$X = \bigcup_{n \in \mathbb{N}} K_n.$$

Lemma 3.2.1.1: Let $\Omega \subseteq \mathbb{C}$ be an open set and let $\mathfrak{B}$ be a basis for the topology on Ω. Then there exists a collection of sets $\{U_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{B}$ such that

1. $\bigcup_{n\in\mathbb{N}} U_n = \Omega$.
2. For every compact $K \subset \Omega$, K intersects only finitely many sets in $\{U_n\}_{n\in\mathbb{N}}$.

Proof: Let $\{K_n\}_{n\in\mathbb{N}} \subset \Omega$ be an exhaustion by compact sets with $K_0 = \varnothing$ and $K_n \subseteq \overset{\circ}{K}_{n+1}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define

$$W_n = \overset{\circ}{K}_{n+1} \setminus K_{n-2}, \quad V_n = K_n \setminus \overset{\circ}{K}_{n-1},$$

where $K_{-1} = \varnothing$. Each W_n is open and each V_n is compact, with $V_n \subseteq W_n$ and

$$\bigcup_{n\in\mathbb{N}} V_n = \Omega.$$

For each $n \in \mathbb{N}$ and each $z \in V_n$, since W_n is open and contains z, there exists $U_{z,n} \in \mathfrak{B}$ such that

$$z \in U_{z,n} \subseteq W_n.$$

The collection

$$\{U_{z,n} : z \in V_n\}$$

is an open cover of the compact set V_n, so by Heine–Borel (Theorem 1.1.3) it admits a finite subcover, there exist finitely many points $z_{n,1}, ..., z_{n,k_n} \in V_n$ such that

$$V_n \subset \bigcup_{i=1}^{k_n} U_{z_{n,i},n} \subseteq W_n.$$

Enumerate all such $U_{z_{n,i},n}$ over $n \in \mathbb{N}$ and $i = 1, ..., k_n$ to obtain a countable collection $\{U_j\}_{j\in\mathbb{N}} \subseteq \mathfrak{B}$. Then

$$\bigcup_{j\in\mathbb{N}} U_j = \Omega,$$

proving Part 1.

For 2, let $K \subset \Omega$ be compact. There exists $N \in \mathbb{N}$ such that

$$K \subset \overset{\circ}{K}_N,$$

so K is disjoint from V_n for all $n > N + 1$. Since each V_n intersects only finitely many U_j, K intersects only finitely many U_j. Thus the collection is locally finite. ☐

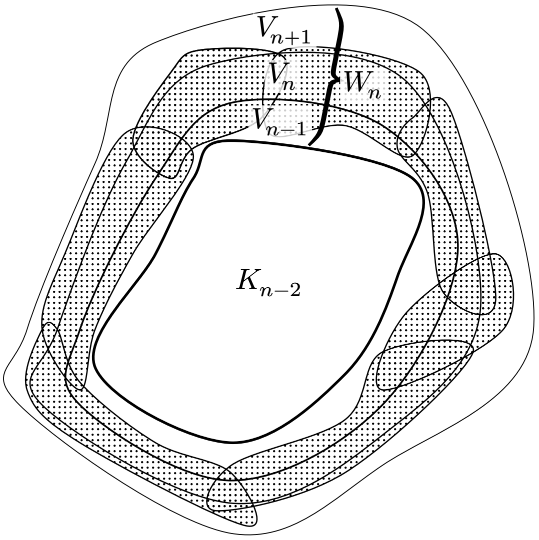

Figure 4: Geometry of the finite subcover of $V_n \subset W_n$ for some $n \in \mathbb{N}$.

Remark: The property of local finiteness of an open collection S in Ω is commonly stated as: for every $z \in \Omega$, there exists an open neighborhood of z that intersects only finitely many sets in S.

This is equivalent to Part 2 in Lemma 3.2.1.1. Indeed, if every point has such a neighborhood, then any compact $K \subset \Omega$ admits a finite subcover of these neighborhoods by Heine–Borel (Theorem 1.1.3), so K intersects finitely many sets in S. Conversely, for any $z \in \Omega$, take an open neighborhood V with $z \in V$ and with relatively compact closure in Ω; then $\overline{V}$ intersects finitely many sets in S, and so does V.

Theorem 3.2.1.5 *(Partition of Unity)*: Let $\Omega \subseteq \mathbb{C}$ be a nonempty open set and let $\{\Omega_k\}_{k\in\mathbb{N}}$ be an open cover of Ω. Then there exists a collection of bump functions $\{\alpha_j\}_{j\in\mathbb{N}} \subseteq C^\infty(\mathbb{C})$, each with compact support in Ω, satisfying:

1. For each $j \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $\operatorname{supp}(\alpha_j) \subseteq \Omega_k$.
2. The collection $\{\operatorname{supp}(\alpha_j)\}_{j\in\mathbb{N}}$ is locally finite.
3. For each $j \in \mathbb{N}$, $0 \leq \alpha_j \leq 1$.
4. $\sum_{j=1}^{\infty} \alpha_j \equiv 1$ on Ω.

Then $\{\alpha_j\}_{j\in\mathbb{N}}$ is called a C^∞ partition of unity subordinate to $\{\Omega_k\}_{k\in\mathbb{N}}$.

Proof: For each $z \in \Omega$ there exists $r_z > 0$ and $k_z \in \mathbb{N}$ such that

$$\overline{D(z, r_z)} \subset \Omega_{k_z}.$$

The collection

$$\{D(z,r) : z \in \Omega \wedge 0 < r < r_z\}$$

is an open basis for Ω. By Lemma 3.2.1.1 there exists a locally finite open cover

$$\left\{D\left(z_j, r_{z_j}\right)\right\}_{j\in\mathbb{N}} \subseteq \mathfrak{B}$$

of Ω with

$$D\left(z_j, r_{z_j}\right) \subset \overline{D\left(z_j, r_{z_j}\right)} \subset \Omega_{k_{z_j}}, \quad \forall j \in \mathbb{N}.$$

Define the standard bump function

$$\theta(z) = \begin{cases} \mathrm{e}^{\frac{1}{|z|^2-1}} & \text{if } |z| < 1 \\ 0 & \text{if } |z| \geq 1. \end{cases}$$

For $\varepsilon > 0$ let

$$\theta_{\varepsilon(z)} = \theta\left(\frac{z}{\varepsilon}\right),$$

which has support $\overline{D(0,\varepsilon)}$. Define

$$\beta_j(z) = \theta_{r_{z_j}}\left(z - z_j\right),$$

so

$$\operatorname{supp}\left(\beta_j\right) = \overline{D\left(z_j, r_{z_j}\right)} \subset \Omega_{k_{z_j}}.$$

By local finiteness of $\left\{D\left(z_j, r_{z_j}\right)\right\}_{j\in\mathbb{N}}$, for each $z \in \Omega$ there exists an open neighborhood V with $z \in V$ intersecting only finitely many $\overline{D\left(z_j, r_{z_j}\right)}$. Thus $\left\{\operatorname{supp}\left(\beta_j\right)\right\}_{j\in\mathbb{N}}$ is locally finite on Ω. Then the sum

$$S(z) = \sum_{j=1}^{\infty} \beta_j(z)$$

defined for $z \in \Omega$ involves only finitely many nonzero terms (by local finiteness) on a neighborhood of every point z. Hence $S \in C^\infty(\Omega)$ and $S(z) > 0$ (since $\left\{D\left(z_j, r_{z_j}\right)\right\}_{j\in\mathbb{N}}$ covers Ω). Define

$$\alpha_j(z) = \frac{\beta_j(z)}{S(z)}, \quad \forall j \in \mathbb{N}.$$

Each $\alpha_j \in C^\infty(\mathbb{C})$ has compact support in Ω, $0 \leq \alpha_j \leq 1$, the supports are locally finite, and

$$\sum_{j=1}^{\infty} \alpha_j(z) = 1$$

for all $z \in \Omega$. Moreover

$$\operatorname{supp}(\alpha_j) \subseteq \Omega_{k_{z_j}},$$

proving subordination. □

Theorem 3.2.1.6 *(Existence of Bump Functions)*: Let $K \subset \mathbb{C}$ be compact and $V \subset \mathbb{C}$ an open neighborhood of K. Then there exists a compactly supported $\varphi \in C^\infty(\mathbb{C})$ such that

$$0 \leq \varphi(z) \leq 1 \quad \forall z \in \mathbb{C},$$

$\operatorname{supp}(\varphi) \subset V$, and $\varphi \equiv 1$ on some open neighborhood of K.

Proof: Let

$$V(K, \varepsilon) = \left\{ z \in \mathbb{C} : \inf_{\zeta \in K} |z - \zeta| < \varepsilon \right\}$$

denote the open ε-neighborhood of K. Since V is an open neighborhood of K, $\exists \varepsilon > 0$ such that

$$K \subset V(K, \varepsilon) \subset V(K, 2\varepsilon) \subset V,$$

where $A \subset B$ means that the closure of A is compact and contained in B.

Define the open sets

$$\Omega_1 = V(K, 2\varepsilon), \quad \Omega_2 = \mathbb{C} \setminus \overline{V(K, \varepsilon)}.$$

Then $\{\Omega_1, \Omega_2\}$ is an open cover of $\mathbb{C}$.

By the Partition of Unity Theorem (Theorem 3.2.1.5), there exist compactly supported functions $\{\alpha_j\}_{j \in \mathbb{N}} \subseteq C^\infty(\mathbb{C})$ forming a partition of unity subordinate to this cover. That is,

$$0 \leq \alpha_j \leq 1, \quad \operatorname{supp}(\alpha_j) \subseteq \Omega_{i_j} \text{ for some } i_j \in \{1, 2\}, \quad \sum_{j=1}^{\infty} \alpha_j \equiv 1 \quad \text{on } \mathbb{C}.$$

Let

$$J = \{ j \in \mathbb{N} : \operatorname{supp}(\alpha_j) \subseteq \Omega_1 \}.$$

Define

$$\varphi(z) = \sum_{j \in J} \alpha_j(z).$$

Then $\varphi \in C^{\infty}(\mathbb{C})$ is compactly supported within Ω_1, and since only finitely many α_j are nonzero on a neighborhood of each point, $\varphi \in C^{\infty}(\mathbb{C})$. Moreover,

$$\operatorname{supp}(\varphi) \subset \Omega_1 \subset V.$$

For $z \in V(K, \varepsilon)$, all functions with support in Ω_2 vanish at z, so

$$\varphi(z) = \sum_{j \in J} \alpha_j(z) = \sum_{j=1}^{\infty} \alpha_j(z) = 1.$$

Hence, $\varphi \equiv 1$ on the open neighborhood $V(K, \varepsilon)$ of K. Outside $V(K, 2\varepsilon)$, all terms with support in Ω_1 vanish, so $\varphi(z) = 0$. Finally, $0 \leq \varphi \leq 1$ everywhere by construction. Thus φ satisfies all required properties. □

3.3 Zeros of a Holomorphic Function

For a region $U \subseteq \mathbb{C}$ and a holomorphic function $f : U \to \mathbb{C}$, a point $z_0 \in U$ is a *zero* of f iff $f(z_0) = 0$. Furthermore, if f has the Taylor expansion at z_0 of

$$a_m (z - z_0)^m + a_{m+1} (z - z_0)^{m+1} + \cdots, \quad m \in \mathbb{N}, a_m \neq 0,$$

then the zero at z_0 has multiplicity m.

We will introduce a fundamental application of Liouville's Theorem (Theorem 3.2.3) below.

Theorem 3.3.1 *(Fundamental Theorem of Algebra)*: Every non-constant polynomial $p(z)$ with complex coefficients has at least one complex zero.

Proof: For the sake of contradiction, suppose that $p(z)$ has no complex zeros. Then the function $f(z) = \frac{1}{p(z)}$ is continuous and entire, because $p(z)$ has no zeros in $\mathbb{C}$. Moreover, as $z \to \infty$, $p(z) \to \infty$, so $f(z) \to 0$, and thus $f(z)$ is bounded. By Liouville's Theorem (Theorem 3.2.3), every bounded entire function is constant. Thus, $f(z)$ is constant, and so $p(z)$ must also be constant. By contradiction, $p(z)$ has at least one complex zero. □

Theorem 3.3.2: Let $U \subseteq \mathbb{C}$ be open and connected, and $f : U \to \mathbb{C}$ be holomorphic over U. Then if the set defined by

$$S = \{z \in U : f(z) = 0\}$$

has an accumulation point in U, then $f \equiv 0$ over U.

Proof: Let $\{z_n\}_{n\in\mathbb{N}}$ be a subset of S and assume it has an accumulation point z_∞ in U. Since f is holomorphic over U, $\exists\varepsilon > 0$ such that f is holomorphic over $D(z_\infty, \varepsilon) \subseteq U$. Then over this disk, f has the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_\infty)^n. \tag{3.3.1}$$

By Definition 1.1.3, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $z_n \in D(z_\infty, \varepsilon)$. Since z_n is a zero of f, $f(z_n) = 0$. Then, by the continuity of f,

$$\lim_{n\to\infty} f(z_n) = f\Big(\lim_{n\to\infty} z_n\Big) = f(z_\infty) = 0.$$

Using this result in comparison to (3.3.1), we get that $a_0 = 0$.

The function

$$f_1(z) = \frac{f(z)}{z - z_\infty}$$

has a Taylor expansion over $D(z_\infty, \varepsilon)$ of

$$f_1(z) = \sum_{n=0}^{\infty} a_{n+1} (z - z_\infty)^n.$$

Let $z = z_n \neq z_\infty$ for some $n > N$. Then f_1 vanishes, leaving

$$0 = a_1 + \mathcal{O}(z_n - z_\infty).$$

Letting $n \to \infty$, $z_n \to z_\infty$, and $a_1 = 0$. Define

$$f_2(z) = \frac{f_1(z)}{z - z_\infty}.$$

Then,

$$f_2(z) = \sum_{n=0}^{\infty} a_{n+2} (z - z_\infty)^n.$$

Similarly, $a_2 = 0$. Letting

$$f_n(z) = \frac{f(z)}{(z - z_\infty)^n},$$

the sequence $\{a_n\}_{n\in\mathbb{Z}_{\geq 0}}$ vanishes, and $f \equiv 0$ on $D(z_\infty, \varepsilon)$.

Let

$$\tilde{S} = \big\{z \in U : \forall n \in \mathbb{Z}_{\geq 0}, f^{(n)}(z) = 0\big\}.$$

For all $z \in D(z_\infty, \varepsilon)$, since $f(z)$ locally vanishes (and has vanishing derivatives as a consequence),

$$D(z_\infty, \varepsilon) \subseteq \tilde{S}.$$

Furthermore, for all $z' \in \tilde{S}$, $\exists \varepsilon' > 0$ such that $f(z)$ has a convergent Taylor series with vanishing coefficients on $D(z', \varepsilon') \subseteq U$. Then $f \equiv 0$ on $D(z', \varepsilon')$. Then for all $z \in D(z', \varepsilon')$, since f is constant at z, it also has vanishing derivatives. It follows that

$$D(z', \varepsilon') \subseteq \tilde{S}.$$

Since every point in $\tilde{S}$ has an open neighborhood also in $\tilde{S}$, $\tilde{S}$ is open.

It is evident that for all $k \in \mathbb{Z}_{\geq 0}$, $f^{(k)}$ is continuous in U by the holomorphy of f. Let

$$S_k = \left\{z \in U : f^{(k)}(z) = 0\right\}.$$

For any sequence $\{\tilde{z}_n\} \in S_k$ converging to some $\tilde{z}_\infty \in U$, by the continuity of f,

$$\lim_{n\to\infty} f^{(k)}(\tilde{z}_n) = f^{(k)}\left(\lim_{n\to\infty} \tilde{z}_n\right) = f^{(k)}(\tilde{z}_\infty) = 0,$$

and therefore $\tilde{z}_\infty \in S_k$. Thus, S_k contains all of its accumulation points in U and is therefore closed in U (if $\tilde{z}_\infty \notin U$, then it is no longer relevant; we are concerned about it being closed within U). Since

$$\tilde{S} = \bigcap_{k \in \mathbb{Z}_{\geq 0}} S_k$$

and each of S_k is closed in U, $\tilde{S}$ is the intersection of closed sets and consequently closed.

Since $\tilde{S}$ is nonempty and clopen in the connected set U, $\tilde{S} = U$ (by Theorem 3.2.1.3). It follows that $f \equiv 0$ on U. □

Remark: This is a trivial property of holomorphic functions that allows for the uniqueness of analytic continuations. It is oftentimes stated in the form below:

Theorem 3.3.3 *(IDENTITY THEOREM)*: Let $U \subseteq \mathbb{C}$ be open and connected, and define $f(z)$ and $g(z)$ to be two holomorphic functions on U. For a set $S \subseteq U$ with an accumulation point in U, if $f \equiv g$ on S, then $f \equiv g$ on U.

Proof: Let $h = f - g$ be holomorphic over U. Since S has an accumulation point in U, and $h \equiv 0$ over S, then by Theorem 3.3.2, $h \equiv 0$ over U. □

Theorem 3.3.4 *(Holomorphic Argument Principle)*: Let $U \subseteq \mathbb{C}$ be a region and $f : U \to \mathbb{C}$ be holomorphic. Let $\gamma \subset U$ be a simple, closed, positively oriented curve that is null-homotopic in U. If f has no zeros on γ, then f has finitely many zeros in the region bounded by γ, and this number, counting multiplicities, is given by

$$k = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{f'(z)}{f(z)} \, \mathrm{d}z.$$

Let Γ be the image of γ under the map $w = f(z)$. Then

$$k = \frac{1}{2\pi} \Delta_{\Gamma} \arg(w),$$

where $\Delta_{\Gamma} \arg(w)$ denotes the total change in argument of w as it traverses Γ.

Proof: Let $z_1, ..., z_n$ be the distinct zeros of f enclosed by γ with the respective multiplicities $k_1, ..., k_n$. Choose disjoint disks $D\left(z_j, \varepsilon_j\right)$ centered at each z_j with radii $\varepsilon_j > 0$, each contained in the interior of γ and avoiding γ. The function

$$\frac{f'(z)}{f(z)}$$

is holomorphic on the domain

$$\operatorname{int}(\gamma) \setminus \bigcup_{j=1}^{n} \overline{D\left(z_j, \varepsilon_j\right)},$$

where $\operatorname{int}(\gamma)$ denotes the interior relative to γ. The oriented boundary of this domain is

$$\gamma^{+} \cup \bigcup_{j=1}^{n} \partial D\left(z_j, \varepsilon_j\right)^{-}.$$

By Cauchy–Goursat (Theorem 3.1.7),

$$\int_{\gamma^{+} \cup \bigcup_{j=1}^{n} \partial D\left(z_j, \varepsilon_j\right)^{-}} \frac{f'(z)}{f(z)} \, \mathrm{d}z = 0,$$

which rearranges to

$$\oint_{\gamma^{+}} \frac{f'(z)}{f(z)} \, \mathrm{d}z = \sum_{j=1}^{n} \oint_{\partial D\left(z_j, \varepsilon_j\right)^{+}} \frac{f'(z)}{f(z)} \, \mathrm{d}z.$$

Near each z_j, express

$$f(z) = \left(z - z_j\right)^{k_j} h_j(z)$$

where h_j is holomorphic and non-vanishing on $D\left(z_j, \varepsilon_j\right)$. Differentiation yields

$$f'(z) = k_j\left(z - z_j\right)^{k_j - 1} h_j(z) + \left(z - z_j\right)^{k_j} h_{j'}(z),$$

and thus

$$\frac{f'(z)}{f(z)} = \frac{k_j}{z - z_j} + \frac{h_{j'}(z)}{h_j(z)}.$$

Since h_j is holomorphic and non-vanishing on $D\left(z_j, \varepsilon_j\right)$, the function $h_{j'}/h_j$ is holomorphic there. By the Cauchy–Goursat Theorem,

$$\oint_{\partial D(z_j, \varepsilon_j)} \frac{h_{j'}(z)}{h_j(z)} \,\mathrm{d}z = 0.$$

The Cauchy–Goursat Formula (Theorem 3.1.8) gives

$$\oint_{\partial D(z_j, \varepsilon_j)} \frac{k_j}{z - z_j} \,\mathrm{d}z = 2\pi \mathrm{i} k_j.$$

Combining results,

$$\oint_{\Gamma} \frac{f'(z)}{f(z)} \,\mathrm{d}z = \sum_{j=1}^{n} 2\pi \mathrm{i} k_j = 2\pi \mathrm{i} k.$$

Finally, parameterize Γ by $w = f(z)$. Then $\mathrm{d}w = f'(z)\,\mathrm{d}z$, and

$$k = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{\mathrm{d}w}{w} = \frac{1}{2\pi \mathrm{i}} \Delta_{\Gamma} \log(w) = \frac{1}{2\pi} \Delta_{\Gamma} \arg(w),$$

which proves the result. □

Thus, one defines the *winding index* (Ind) to quantify how many times a closed curve winds counterclockwise around a given point in the complex plane. Formally, if $\gamma = \gamma([0,1])$ is a counterclockwise-oriented closed curve and z is a point satisfying $z \notin \gamma$, then

$$\mathrm{Ind}_{\Gamma}(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma} \frac{\mathrm{d}\zeta}{\zeta - z} = \frac{1}{2\pi \mathrm{i}} \int_0^1 \frac{\gamma'(t)\,\mathrm{d}t}{\gamma(t) - z}.$$

Theorem 3.3.5: Let $\{f_n(z)\}$ be a sequence of holomorphic functions on the open set $U \subseteq \mathbb{C}$ that uniformly converges to $f(z)$ on every compact subset of U. If $\forall n \in \mathbb{N}$, $f_n(z)$ has no zeros in U, then f is either identically 0 or has no zeros in U.

Proof: By the holomorphy of $f_n(z)$, for any simple closed rectifiable curve $\gamma \subset U$ (whose interior is a subset of U), by the Cauchy–Goursat Theorem (Theorem 3.1.7),

$$\oint_{\Gamma} f_n(\zeta)\,\mathrm{d}\zeta = 0.$$

Since γ is a subset of any compact subset of U, $\{f_n(\zeta)\}$ uniformly converges on γ, and by Theorem 2.3.6,

$$\lim_{n\to\infty} \oint_{\Gamma} f_n(\zeta)\,\mathrm{d}\zeta = \oint_{\Gamma} \lim_{n\to\infty} f_n(\zeta)\,\mathrm{d}\zeta = \oint_{\Gamma} f(\zeta)\,\mathrm{d}\zeta = 0. \quad (3.3.2)$$

Then by Morera's Theorem (Theorem 3.2.4), $f(z)$ is holomorphic, and $f'(z)$ is holomorphic. We aim to show that $f_n'(z) \rightrightarrows f'(z)$.

Let $K \subset U$ be arbitrary and compact and $V \supset K$ be open and relatively compact in U. Since $\{f_n'(z)\}$ is holomorphic, by Corollary 3.2.5.1, there exists a finite constant $c > 0$ such that

$$\lim_{n\to\infty} \sup_{z\in K} |f_n'(z) - f'(z)| \le c \lim_{n\to\infty} \sup_{z\in V} |f_n(z) - f(z)|.$$

By the definition of uniform convergence, the right-hand side approaches 0, and $\{f_n'(z)\}$ is then uniformly convergent to $f'(z)$ by the same reasoning.

Through the proof of Theorem 3.3.2, if $f \not\equiv 0$ over U, then the zeros of f do not have an accumulation point in U and are therefore discrete. In this case, let $\gamma \subset U$ be a curve that does not pass through the zeros of f. Since each function in the sequence f_n does not contain zeros in U, by the Argument Principle (Theorem 3.3.4),

$$\lim_{n\to\infty} \oint_{\Gamma} \frac{f_n'(z)}{f_n(z)}\,\mathrm{d}z = 0. \quad (3.3.3)$$

Since f and f' are holomorphic over γ, by Theorem 1.2.13, there exists a finite value $M > 0$ such that $\forall z \in \gamma$, $\max\{|f(z)|, |f'(z)|\} < M$.

Since γ does not pass through the zeros of f, $\exists \lambda > 0$ such that $\forall z \in \gamma$, $|f(z)| > \lambda$. By the uniform convergence of $\{f_n(z)\}$, $\exists N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \frac{\lambda}{2}, \quad \forall n > N, \forall z \in \gamma.$$

Then $|f_n(z)| > \frac{\lambda}{2}$ on γ. Hence, $\frac{1}{f_n(z)}$ and its limit are uniformly bounded;

$$\left|\frac{1}{f(z)}\right| < \frac{1}{\lambda}, \quad \left|\frac{1}{f_n(z)}\right| < \frac{2}{\lambda}, \quad \forall z \in \gamma, \forall n > N.$$

$$\begin{aligned}\left|\frac{f'}{f} - \frac{f_n'}{f_n}\right| &= \left|\frac{f' f_n - f_n' f}{f_n f}\right| \\ &< 2\frac{|f' f_n - f' f| + |f' f - f_n' f|}{\lambda^2} \\ &< \frac{2M}{\lambda^2} \cdot (|f_n - f| + |f' - f_n'|).\end{aligned}$$

$$\begin{aligned}\sup_{z\in\gamma}\left|\frac{f'(z)}{f(z)} - \frac{f_n'(z)}{f_n(z)}\right| &\leq \frac{2M}{\lambda^2}\left(\sup_{z\in\gamma}|f_n(z) - f(z)| + \sup_{z\in\gamma}|f'(z) - f_n'(z)|\right) \\ \lim_{n\to\infty}\sup_{z\in\gamma}\left|\frac{f'(z)}{f(z)} - \frac{f_n'(z)}{f_n(z)}\right| &\leq \frac{2M}{\lambda^2}\left(\lim_{n\to\infty}\sup_{z\in\gamma}|f_n(z) - f(z)| + |f'(z) - f_n'(z)|\right) \\ &= 0.\end{aligned}$$

Therefore, $\frac{f'(z)}{f(z)}$ is uniformly convergent on γ. By Theorem 2.3.6, we can pass the limit through the integral in (3.3.3). Then,

$$\lim_{n\to\infty}\oint_\gamma \frac{f_n'(z)}{f_n(z)}\,\mathrm{d}z = \oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = 0.$$

By the Argument Principle (Theorem 3.3.4), $f(z)$ has no zeros in the interior of γ. Since γ was arbitrarily chosen, either $f(z) \equiv 0$ on U or has no zeros in U. □

Theorem 3.3.6 *(Rouché)*: Let $U \subseteq \mathbb{C}$ be open and f, g be two holomorphic functions over U. Let $\gamma \subset U$ be a simple, closed, rectifiable curve, and for all $z \in \gamma$

$$|f(z) - g(z)| < |f(z)|. \tag{3.3.4}$$

Then f and g have the same number of zeros enclosed by γ and do not vanish on γ.

Proof: It is obvious that $g(z)$ has no zeros on γ. Otherwise, $\exists z_0 \in \gamma$ such that $g(z_0) = 0$, implying that $|f(z_0)| < |f(z_0)|$ which is impossible. Similarly, $f(z)$ has no zeros on γ, since $|g(z)| < 0$ is an impossibility.

Let k_f and k_g denote the number of zeros of f and g enclosed by γ, respectively. By the Argument Principle (Theorem 3.3.4),

$$\begin{aligned} k_g - k_f &= \oint_\Gamma \frac{g'(z)}{g(z)}\,\mathrm{d}z - \oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z \\ &= \oint_\gamma \frac{g'(z)f(z) - f'(z)g(z)}{g(z)f(z)}\,\mathrm{d}z = \oint_\gamma \frac{\left(\frac{g(z)}{f(z)}\right)'}{\frac{g(z)}{f(z)}}\,\mathrm{d}z. \end{aligned}$$

Let $w = h(z) = \frac{g(z)}{f(z)}$ with $\Gamma = h(\gamma)$. Then,

$$k_g - k_f = \oint_\Gamma \frac{\mathrm{d}w}{w}.$$

From (3.3.4), by dividing both sides by $f(z)$, we obtain $|w - 1| < 1$. Then Γ lies in the open disk $D(1, 1)$, which will never intersect or enclose 0. Then by Lemma 3.1.2,

$$k_g - k_f = \oint_\Gamma \frac{\mathrm{d}w}{w} = 0,$$

as desired. □

By the Fundamental Theorem of Algebra (Theorem 3.3.1), any polynomial in the form $p(z) = \sum_{k=0}^{n} a_k z^k$ ($n \in \mathbb{N}, a_n \neq 0, a_k \in \mathbb{C}$ where $k = 1, \dots n$) has at least one complex zero. Consider the function $q(z) = a_n z^n$, with a zero at $z = 0$ with multiplicity n. By Rouché's Theorem (Theorem 3.3.6), since $\exists R \in \mathbb{R}$ such that $|q(z) - p(z)| = \left|\sum_{k=0}^{n-1} a_k z^k\right| < |a_n z^n|$ over $|z| = R$, p and q have the same number of zeros, counting multiplicity.

Theorem 3.3.7: Let $U \subseteq \mathbb{C}$ be open and connected, and $f(z)$ be holomorphic and non-constant on U.

If $z_0 \in U$ and $w_0 = f(z_0)$, and the multiplicity of the zero at z_0 of $f - w_0$ is m, then for all $\rho > 0$ such that $f - w_0$ is non-vanishing on $\overline{D(z_0, \rho)} \setminus \{z_0\}$, $\exists \delta > 0$ such that $\forall \xi \in D(w_0, \delta)$, $f - \xi$ has m zeros in $D(z_0, \rho)$, counting multiplicity.

Proof: The zero at z_0 is isolated by Theorem 3.3.2. Furthermore, $|f - w_0|$ is continuous on $\partial D(z_0, \rho)$ and attains a positive infimum δ. In other words, on this set, $|f - w_0| \geq \delta$. Hence, $\forall \xi \in D(w_0, \delta)$, we have $|\xi - w_0| < \delta \leq |f(z) - w_0|$ for any $z \in \partial D(z_0, \rho)$.

By Rouché's Theorem, since $|(f(z) - w_0) - (f(z) - \xi)| < |f(z) - w_0|$, it follows that $f - \xi$ and $f - w_0$ have the same number of zeros in $D(z_0, \rho)$. □

We also have the following generalization of Theorem 3.3.5, which is a heuristic restatement of Theorem 3.3.7:

Theorem 3.3.8 *(Hurwitz)*: Let $U \subseteq \mathbb{C}$ be an open and connected set, and suppose $\{f_n(z)\}_{n\in\mathbb{N}}$ is a holomorphic function sequence that uniformly converges to a non-constant function $f(z)$ on all compact sets of U.

If $z_0 \in U$ and $w_0 = f(z_0)$, and the multiplicity of the zero at z_0 of $f - w_0$ is m, then for all $\rho > 0$ such that $f - w_0$ is non-vanishing on $\overline{D(z_0, \rho)} \setminus \{z_0\}$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $f_n - w_0$ has m zeros in $D(z_0, \rho)$, counting multiplicity.

Proof: The zero at z_0 is isolated by Theorem 3.3.2. Furthermore, $|f - w_0|$ is continuous on $\partial D(z_0, \rho)$ and attains a positive infimum δ. In other words, on this set, $|f - w_0| \geq \delta$. By uniform convergence, $\exists N \in \mathbb{N}$ such that $\forall n > N$, we have $|f(z) - f_n(z)| < \delta \leq |f(z) - w_0|$ for any $z \in \partial D(z_0, \rho)$.

By Rouché's Theorem (Theorem 3.3.6), since

$$|(f(z) - w_0) - (f_n(z) - w_0)| < |f(z) - w_0|,$$

it follows that $f_n - w_0$ and $f - w_0$ have the same number of zeros in $D(z_0, \rho)$. ☐

3.4 Further Properties of Holomorphic Functions

A useful corollary of Theorem 3.1.8 is the Maximum Modulus Principle.

Before the theorem, we first introduce the mean-value property of holomorphic functions.

Lemma 3.4.1: Let $U \subseteq \mathbb{C}$ be open and simply connected, and let $f : U \to \mathbb{C}$ be holomorphic. Then $\forall z \in U$ and $\forall \varepsilon > 0$ such that $\overline{D(z, \varepsilon)} \subset U$, $f(z)$ is the average of $f(\zeta)$ where $\zeta \in D(z, \varepsilon)$ is uniform. In other words,

$$f(z) = \frac{1}{2\pi\varepsilon} \oint_{\partial D(z,\varepsilon)} f(\zeta)|\mathrm{d}\zeta|.$$

Proof: By the Cauchy–Goursat Formula (Theorem 3.1.8),

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta = \frac{1}{2\pi} \int_0^{2\pi} f(z + \varepsilon \mathrm{e}^{\mathrm{i}t}) \, \mathrm{d}t.$$

Observe that

$$\begin{aligned} f(z) = \frac{1}{2\pi\varepsilon} \oint_{\partial D(z,\varepsilon)} f(\zeta)|\mathrm{d}\zeta| &= \frac{1}{2\pi\varepsilon} \int_0^{2\pi} f(z + \varepsilon \mathrm{e}^{\mathrm{i}t}) |\mathrm{i}\varepsilon \mathrm{e}^{\mathrm{i}t} \, \mathrm{d}t| \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z + \varepsilon \mathrm{e}^{\mathrm{i}t}) \, \mathrm{d}t, \end{aligned}$$

and the conclusion follows. □

Since the real and imaginary parts of holomorphic functions are real-valued harmonic functions, they also satisfy the mean-value property. Furthermore, if a real continuous function satisfies the mean-value property, it is harmonic (to be proved in Theorem 3.6.2.1). This equivalence allows for the alternative definition of harmonic functions.

Theorem 3.4.1 *(Maximum Modulus Principle)*: Let $f(z)$ be holomorphic on an open connected region $U \subseteq \mathbb{C}$. If $\exists z_0 \in U$ and an open neighborhood $V \subseteq U$ of z_0 such that $\forall z \in V$, $|f(z_0)| \geq |f(z)|$, then f is a constant function on U.

Proof: Assume that z_0 exists. We will first prove that the set

$$S = \{z : f(z) = f(z_0), z \in V\}$$

is all of V. This is equivalent to proving that S is nonempty, open, and closed in V.

Since $z_0 \in S$, the first condition is satisfied (nonemptiness). For any sequence $\{z_n\} \in S$ converging to some $z_\infty \in V$, by the continuity of f,

$$\lim_{n\to\infty} f(z_n) = f\left(\lim_{n\to\infty} z_n\right) = f(z_\infty) = f(z_0),$$

and $z_\infty \in S$. Thus, S contains all of its accumulation points in V and is therefore closed (if $z_\infty \notin V$, then it is no longer relevant; we are concerned with its relative closedness in V).

Since $S \subseteq V$ and V are both open, $\forall z \in S$, $\exists \lambda > 0$ such that $D(z, \lambda) \subseteq V$. By Lemma 3.4.1, $\forall 0 < \varepsilon < \lambda$,

$$\begin{aligned} |f(z)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(z + \varepsilon \mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t\right| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(z + \varepsilon \mathrm{e}^{\mathrm{i}t})\right|\mathrm{d}t \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z)|\,\mathrm{d}t = |f(z)|. \end{aligned}$$

It follows that all inequalities above are equalities, or that

$$\begin{aligned} |f(z)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(z + \varepsilon \mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t\right| = \frac{1}{2\pi}\int_0^{2\pi} \left|f(z + \varepsilon \mathrm{e}^{\mathrm{i}t})\right|\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} |f(z)|\,\mathrm{d}t = |f(z)|. \end{aligned}$$

From the equality of the last two integrals,

$$\int_0^{2\pi} \left[|f(z)| - \left|f(z + \varepsilon e^{it})\right|\right] \mathrm{d}t = 0.$$

Since this integrand is strictly non-negative, we have equality. Thus, $\forall z \in S$, $\exists \lambda > 0$ such that $D(z, \lambda) \subseteq S$. In other words, every $z \in S$ has an open neighborhood that also lies in S. Therefore, S is open and $S = V$ as it is a nonempty clopen subset. Since V is nonempty and open, it has an accumulation point in U. It follows that $f(z) \equiv f(z_0)$ over U by the Identity Theorem (Theorem 3.3.3). $\square$

Remark: If f is holomorphic and non-constant on an open region $U \subseteq \mathbb{C}$, then for any compact set $K \subset U$, the maximum of f in K lies on ∂K. Otherwise, f would attain a maximum at some $z \in \overset{\circ}{K}$, and contradict the statement of Theorem 3.4.1 under the assumption of being non-constant.

A similar theorem exists for real-valued harmonic functions. The proof follows in the same way as the one for holomorphic functions. We will state it formally below.

Theorem 3.4.2: Let $U \subseteq \mathbb{C}$ be open and connected and let $f : U \to \mathbb{R}$ be harmonic. Suppose that $\exists z_0 \in U$ and a neighborhood $V \subseteq U$ of z_0 such that either

$$f(z) \geq f(z_0) \quad \forall z \in V \quad \text{or} \quad f(z_0) \geq f(z) \quad \forall z \in V.$$

Then f is constant on U.

By nature of the proof, the result holds for any continuous function satisfying the mean-value property.

3.5 The Group of Holomorphic Automorphisms on the Unit Disk

The following important result can be directly obtained from the Maximum Modulus Principle.

Lemma 3.5.1 *(Schwarz)*: If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$, then

$$|f(z)| \leq |z|, \quad |f'(0)| \leq 1.$$

Any one of the inequalities becomes equalities iff $f(z)$ is in the form of $ze^{i\tau}$, where $\tau \in \mathbb{R}$. In other words, f is a pure rotation.

Proof: Define the auxiliary function

$$g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0. \end{cases}$$

Because $\lim_{z \to 0} \frac{f(z)}{z} = f'(0)$, $g(z)$ is holomorphic on $\mathbb{D}$. Since f is an automorphism on the open disk, $\forall |z| < 1$, $|f(z)| < 1$. By the Maximum Modulus Principle (Theorem 3.4.1), $\forall 0 < \varepsilon < 1$, $\forall z \in D(0, \varepsilon)$,

$$|g(z)| \leq \max_{z_\varepsilon \in \partial D(0,\varepsilon)} \frac{|f(z_\varepsilon)|}{\varepsilon} < \frac{1}{\varepsilon}.$$

As $\varepsilon \to 1^-$, we obtain that $\forall z \in \mathbb{D}$, $|g(z)| \leq 1$, or that $|f(z)| \leq |z|$. Let $z = 0$. Then we get $|g(0)| = |f'(0)| \leq 1$.

For the sake of the equality condition, assume $|f(z)| = |z|$. Then $|g(z)| \equiv 1$ on the unit open disk. By Theorem 3.4.1, $g(z) = \mathrm{e}^{\mathrm{i}\tau}$ where $\tau \in \mathbb{R}$ and $f(z) = z\mathrm{e}^{\mathrm{i}\tau}$ on $\mathbb{D}$.

Next, assume only that $|f'(0)| = 1$. It follows that $|g(0)| = 1$. Since $|g(z)| \leq 1$ for all $z \in \mathbb{D}$, it follows from Theorem 3.4.1 that g is constant with magnitude 1, or in the form of $\mathrm{e}^{\mathrm{i}\tau}$, where $\tau \in \mathbb{R}$ is a constant. Consequently, $f(z) = z\mathrm{e}^{\mathrm{i}\tau}$. $\square$

To discuss the main topic of this section, we will first introduce the concept of a *group*.

Definition 3.5.1 *(Group)*: A group is a nonempty set G and a binary operation (we will denote this as $*$) satisfying the four *group axioms*:

- *Closure*: $\forall a, b \in G$, $a * b \in G$.
- *Associativity*: $\forall a, b, c \in G$, $(a * b) * c = a * (b * c)$.
- *Identity Element*: $\exists e \in G$ such that $\forall a \in G, a * e = e * a = a$. Note that e is unique; if $e, f \in G$ were both identity elements, then $e * f = f * e = e = f$, and are equal.
- *Inverse Element*: $\forall a \in G, \exists a^{-1} \in G$ such that $a * a^{-1} = e = a^{-1} * a$, where e is the identity element. Note that a^{-1} is unique. Assume b, c were both inverses of a. Then, $b = b * e = b * (a * c) = (b * a) * c = c$, and are equal.

A *subgroup* H of G is a subset of G that is also a group under the same operation as G. This relationship is denoted by $H \leq G$ or $H < G$ for *proper subgroups*.

Group operations are not necessarily commutative. In the case that they are, (specifically if $a, b \in G \implies a * b = b * a$), then G is an *abelian group*.

If $U \subseteq \mathbb{C}$ is connected and $f : U \to U$ is holomorphic on U and bijective, f is a *holomorphic automorphism* on U. The *group of holomorphic automorphisms* on U is denoted by $\mathrm{Aut}(U)$, which is the set of all holomorphic automorphisms such as f, with the operation of composition $(f \circ g)$.

First we will show that $\forall a \in \mathbb{D}$,

$$\varphi_a(z) = \frac{z-a}{1-\overline{a}z} \in \mathrm{Aut}(\mathbb{D}). \tag{3.5.1}$$

Firstly, the function is holomorphic on $\mathbb{D}$ as $|z| \leq 1$, $|\overline{a}| < 1$, the denominator never vanishes. Additionally, $\varphi_a(a) = 0$.

First, we will observe the image of $\partial\mathbb{D}$. Let $|z| = 1$. Then,

$$|\varphi_a(z)| = \left|\frac{1}{z}\right| \left|\frac{z-a}{\frac{1}{z}-\overline{a}}\right| = \left|\frac{z-a}{\overline{z}-\overline{a}}\right| = 1.$$

Therefore, the image of $\partial\mathbb{D}$ lies on $\partial\mathbb{D}$, and since f is holomorphic and non-constant, by the Maximum Modulus Principle (Theorem 3.4.1), for any $|z| < 1$, $|\varphi_a(z)| < 1$. Therefore, f maps $\mathbb{D}$ to $\mathbb{D}$. We next aim to show that $f : \mathbb{D} \to \mathbb{D}$ is bijective.

Let us first confirm injectivity. For all $z_1, z_2 \in \mathbb{D}$, we will observe when

$$\frac{z_1 - a}{1 - \overline{a}z_1} = \frac{z_2 - a}{1 - \overline{a}z_2}$$

is satisfied. It follows that

$$\begin{gathered} (z_1 - a)(1 - \overline{a}z_2) = (z_2 - a)(1 - \overline{a}z_1), \\ z_1 - a - \overline{a}z_1z_2 + |a|^2 z_2 = z_2 - a - \overline{a}z_1z_2 + |a|^2 z_1. \end{gathered}$$

Then,

$$|a|^2(z_2 - z_1) = z_2 - z_1 \iff (|a|^2 - 1)(z_2 - z_1) = 0.$$

Since $|a| < 1$, then $|a|^2 - 1 \neq 0$, and we get $z_2 - z_1 = 0$. This proves the univalence of $\varphi_a(z)$.

Next, we will solve for the inverse of φ_a. Let $z = \varphi_a(w) = \frac{w-a}{1-\overline{a}w}$. Then,

$$z - \overline{a}zw = w - a \iff w = \frac{z+a}{1+\overline{a}z}. \tag{3.5.2}$$

It follows that $\varphi_{-a} = (\varphi_a)^{-1}$. Thus φ_a is surjective and a bijective automorphism. It follows that (3.5.1) is true. Functions in the form of φ_a (where $a \in \mathbb{D}$) are known as *Möbius transformations*, and the group of all such transformations is known as the *Möbius transformation group on the unit disk*, which is a subgroup of $\mathrm{Aut}(\mathbb{D})$. Functions in the form of $\rho_\tau(z) = z\mathrm{e}^{\mathrm{i}\tau}$, where $\tau \in \mathbb{R}$ is constant, form a group known as the *rotation group*, which is also a subgroup of $\mathrm{Aut}(\mathbb{D})$.

Theorem 3.5.1 *(The Holomorphic Automorphism Group on $\mathbb{D}$)*: $\forall f \in \mathrm{Aut}(\mathbb{D})$, f is a composition between a Möbius transformation and a rotation. In other words, $\exists |a| < 1$ and $\exists \tau \in \mathbb{R}$ such that

$$f(z) = \varphi_a \circ \rho_\tau(z).$$

Moreover, all such functions are in $\mathrm{Aut}(\mathbb{D})$.

Proof: Define the auxiliary function

$$\psi(z) = \varphi_{f(0)} \circ f(z).$$

It follows that $\psi \in \mathrm{Aut}(\mathbb{D})$. Furthermore,

$$\psi(0) = \varphi_{f(0)} \circ f(0) = 0.$$

By the Schwarz Lemma (Lemma 3.5.1), $|\psi'(0)| \leq 1$. Since $\psi^{-1} \in \mathrm{Aut}(\mathbb{D})$ with $\psi^{-1}(0) = 0$, $\left|(\psi^{-1})'(0)\right| \leq 1$. Then,

$$\left|(\psi^{-1})'(0)\right| = \left|\frac{1}{\psi'(\psi^{-1}(0))}\right| = \left|\frac{1}{\psi'(0)}\right| \leq 1.$$

Then, $|\psi'(0)| = 1$, and by the equality statement of Lemma 3.5.1,

$$\psi(z) = z\mathrm{e}^{\mathrm{i}\tau} = \rho_\tau(z)$$

for some constant $\tau \in \mathbb{R}$, and

$$f(z) = \varphi_{f(0)}^{-1} \circ \rho_\tau(z).$$

By (3.5.2), it follows that

$$f(z) = \varphi_{-f(0)} \circ \rho_\tau(z).$$

□

As a direct consequence of Theorem 3.5.1, we have the following result:

Lemma 3.5.2 *(Schwarz–Pick)*: Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. For all $z_1, z_2 \in \mathbb{D}$, let $w_1 = f(z_1)$ and $w_2 = f(z_2)$. Then,

$$\left|\frac{w_1 - w_2}{1 - w_1\overline{w_2}}\right| \leq \left|\frac{z_1 - z_2}{1 - z_1\overline{z_2}}\right|. \tag{3.5.3}$$

and

$$\frac{|\mathrm{d}w|}{1-|w|^2} \leq \frac{|\mathrm{d}z|}{1-|z|^2}. \tag{3.5.4}$$

The equalities hold iff $f \in \mathrm{Aut}(\mathbb{D})$.

Proof: Let

$$\varphi_{-z_1}(z) = \frac{z+z_1}{1+\overline{z_1}z} \in \mathrm{Aut}(\mathbb{D}), \quad \varphi_{w_1}(z) = \frac{z-w_1}{1-\overline{w_1}z} \in \mathrm{Aut}(\mathbb{D}).$$

It follows that

$$\varphi_{w_1} \circ f \circ \varphi_{-z_1}(0) = \varphi_{w_1}(w_1) = 0.$$

Then by the Schwarz Lemma (Lemma 3.5.1), for $z \in \mathbb{D}$,

$$\left|\varphi_{w_1} \circ f \circ \varphi_{-z_1}(z)\right| \leq |z|.$$

Let $z_2 = \varphi_{-z_1}(z)$. Then,

$$\left|\varphi_{w_1} \circ f(z_2)\right| \leq \left|\varphi_{z_1}(z_2)\right| \iff \left|\varphi_{w_1}(w_2)\right| \leq \left|\varphi_{z_1}(z_2)\right|,$$

confirming (3.5.3). By the second statement of the Schwarz Lemma (Lemma 3.5.1), $\left|\left(\varphi_{w_1} \circ f \circ \varphi_{-z_1}\right)'(0)\right| \leq 1$.

By the chain rule,

$$\left|\varphi_{(w_1)'}(w_1) f'(z_1) \varphi_{(-z_1)'}(0)\right| \leq 1.$$

Let us now calculate the derivatives of φ_{w_1} and φ_{-z_1}. By the quotient rule,

$$\varphi'_{w_1}(z) = \frac{1-\overline{w_1}w_1}{(1-\overline{w_1}z)^2}, \quad \varphi'_{w_1}(w_1) = \frac{1}{1-\overline{w_1}w_1},$$

and

$$\varphi'_{-z_1}(z) = \frac{1-\overline{z_1}z_1}{(1+\overline{z_1}z)^2}, \quad \varphi'_{-z_1}(0) = 1-\overline{z_1}z_1.$$

Since both derivatives are positive,

$$|f'(z_1)| \leq \frac{1-\overline{w_1}w_1}{1-\overline{z_1}z_1}.$$

Since $z_1 \in \mathbb{D}$ is arbitrary, it follows that

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| \leq \frac{1-\overline{w}w}{1-\overline{z}z} \iff \frac{|\mathrm{d}w|}{1-\overline{w}w} \leq \frac{|\mathrm{d}z|}{1-\overline{z}z}. \tag{3.5.5}$$

By the Schwarz Lemma (Lemma 3.5.1), under the equality condition that

$$\left|\varphi_{(w_1)'}(w_1)f'(z_1)\varphi_{(-z_1)'}(0)\right| = 1,$$

we have that

$$\varphi_{w_1} \circ f \circ \varphi_{-z_1} = \mathrm{e}^{\mathrm{i}\tau},$$

where $\tau \in \mathbb{R}$ is constant. It follows that

$$f = \varphi_{-w_1} \circ \mathrm{e}^{\mathrm{i}\tau} \circ \varphi_{z_1} \in \mathrm{Aut}(\mathbb{D}). \qquad \square$$

Remark: In Section 8, we will introduce the *hyperbolic metric* on $\mathbb{D}$, defined as

$$\mathrm{d}s^2 = \frac{4|\mathrm{d}z|^2}{\left(1 - |z|^2\right)^2}.$$

From (3.5.5), we get that the hyperbolic metric does in.notcrease under a holomorphic mapping of $\mathbb{D}$ to itself. This metric is invariant (the equality condition) under all functions in $\mathrm{Aut}(\mathbb{D})$. This gives a geometric explanation for Lemma 3.5.1.

3.6 Alternative Integral Formulas

As in the Cauchy Integral Formula (Theorem 3.1.8), we can write holomorphic functions in terms of an integral representation. We define the *Cauchy kernel* to be

$$H(\zeta, z) = \frac{1}{2\pi\mathrm{i}(\zeta - z)}.$$

Then Theorem 3.1.8 can be written as

$$f(z) = \oint_{\partial U} f(\zeta)H(\zeta, z)\,\mathrm{d}\zeta.$$

There also exist other integral formulas for functions, varying in the kernel of the expression.

Let $\Phi : \mathbb{D} \to \mathbb{R}$ be harmonic such that Φ is continuous on $\overline{\mathbb{D}}$. By the mean-value property introduced in Lemma 3.4.1, we have

$$\Phi(0) = \frac{1}{2\pi}\int_0^{2\pi} \Phi\left(\rho\mathrm{e}^{\mathrm{i}t}\right)\mathrm{d}t,$$

where $0 < \rho < 1$. By the uniform continuity of Φ on $\overline{\mathbb{D}}$ (Theorem 1.2.15), $\forall\varepsilon > 0$, $\exists\delta > 0$ such that for all $\rho \in \left(\frac{1}{2}, 1\right)$ satisfying $1 - \rho < \delta$ and all $t \in [0, 2\pi]$,

$$\left|\Phi(\mathrm{e}^{\mathrm{i}t}) - \Phi(\rho\mathrm{e}^{\mathrm{i}t})\right| < \varepsilon.$$

It then follows that

$$\left|\frac{1}{2\pi}\int_0^{2\pi} \Phi(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t - \frac{1}{2\pi}\int_0^{2\pi} \Phi(\rho\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t\right| < \varepsilon.$$

Hence,

$$\lim_{\rho\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} \Phi(\rho\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \Phi(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t = \Phi(0). \tag{3.6.1}$$

Let $z \in \mathbb{D}$ and notice that

$$\varphi_z(\zeta) = \frac{\zeta - z}{1 - \overline{z}\zeta} \in \mathrm{Aut}(\mathbb{D})$$

maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$ bijectively. Let u be harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Then $u \circ \varphi_{-z}$ is also harmonic on $\mathbb{D}$, and by (3.6.1),

$$u(z) = u \circ \varphi_{-z}(0) = \frac{1}{2\pi}\int_0^{2\pi} u \circ \varphi_{-z}\big(\mathrm{e}^{\mathrm{i}\psi}\big)\,\mathrm{d}\psi.$$

By the univalence of φ_z, let $\mathrm{e}^{\mathrm{i}\psi} = \varphi_z\big(\mathrm{e}^{\mathrm{i}t}\big)$. It follows that

$$\begin{aligned}
\mathrm{i}\mathrm{e}^{\mathrm{i}\psi}\,\mathrm{d}\psi &= \mathrm{i}\frac{1 - \overline{z}z}{\big(1 - \overline{z}\mathrm{e}^{\mathrm{i}t}\big)^2}\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t \\
\mathrm{d}\psi &= \frac{1 - \overline{z}z}{\big(1 - \overline{z}\mathrm{e}^{\mathrm{i}t}\big)^2}\frac{1 - \overline{z}\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t} - z}\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t \\
&= \frac{1 - |z|^2}{\left|1 - \overline{z}\mathrm{e}^{\mathrm{i}t}\right|^2}\,\mathrm{d}t.
\end{aligned} \tag{3.6.2}$$

Then from (3.6.1),

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\mathrm{e}^{\mathrm{i}t})\frac{1 - |z|^2}{\left|1 - \overline{z}\mathrm{e}^{\mathrm{i}t}\right|^2}\,\mathrm{d}t.$$

Let

$$P(\zeta, z) = \frac{1 - |z|^2}{2\pi|1 - \overline{z}\zeta|^2} = \frac{1 - |z|^2}{2\pi|\zeta - z|^2},$$

known as the *Poisson kernel*. Then,

$$u(z) = \int_0^{2\pi} u(\zeta) P(\zeta, z) \,\mathrm{d}t, \qquad |z| < 1, \tag{3.6.3}$$

where $\zeta = \mathrm{e}^{\mathrm{i}t}$. (3.6.3) is also known as the *Poisson Integral Formula.*

For all $z \in D(0, R)$, where $R > 0$, we can apply the transformation

$$\tilde{\varphi}_z(\zeta) = R\varphi_{\frac{z}{R}}\left(\frac{\zeta}{R}\right)$$

to extend the automorphism to $\overline{D(0, R)}$. Let u instead be harmonic on $D(0, R)$ and continuous on $\overline{D(0, R)}$. Then,

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u\big(R\mathrm{e}^{\mathrm{i}\psi}\big) \,\mathrm{d}\psi.$$

It follows that $u \circ \tilde{\varphi}_{-z}$ is also harmonic on $D(0, R)$ with

$$u(z) = u \circ \tilde{\varphi}_{-z}(0),$$

and from the bijectivity of $R\mathrm{e}^{\mathrm{i}\psi} = \tilde{\varphi}_z\big(R\mathrm{e}^{\mathrm{i}t}\big)$,

$$\begin{aligned}
\mathrm{d}\psi &= \frac{1 - \frac{|z|^2}{R^2}}{\left(1 - \frac{\overline{z}}{R}\mathrm{e}^{\mathrm{i}t}\right)^2} \mathrm{e}^{\mathrm{i}t}\mathrm{e}^{-\mathrm{i}\psi} \,\mathrm{d}t \\
&= \frac{1 - \frac{|z|^2}{R^2}}{\left(1 - \frac{\overline{z}}{R}\mathrm{e}^{\mathrm{i}t}\right)^2} \frac{1 - \frac{\overline{z}}{R}\mathrm{e}^{\mathrm{i}t}}{1 - \frac{z}{R}\mathrm{e}^{-\mathrm{i}t}} \,\mathrm{d}t \\
&= \frac{R^2 - |z|^2}{\left|R\mathrm{e}^{\mathrm{i}t} - z\right|^2} \,\mathrm{d}t.
\end{aligned} \tag{3.6.4}$$

Then because $\tilde{\varphi}_z^{-1} = \tilde{\varphi}_{-z}$,

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u\big(R\mathrm{e}^{\mathrm{i}t}\big) \frac{R^2 - |z|^2}{\left|R\mathrm{e}^{\mathrm{i}t} - z\right|^2} \,\mathrm{d}t.$$

The expression

$$P(\zeta, z) = \frac{|\zeta|^2 - |z|^2}{2\pi|\zeta - z|^2}$$

is a general form of the Poisson kernel. Then with $\zeta = R\mathrm{e}^{\mathrm{i}t}$,

$$u(z) = \int_0^{2\pi} u(\zeta) P(\zeta, z) \,\mathrm{d}t. \tag{3.6.5}$$

The Poisson kernel can also be rewritten as

$$\begin{aligned} P(\zeta, z) &= \frac{|\zeta|^2 - |z|^2}{2\pi(\zeta - z)\left(\overline{\zeta} - \overline{z}\right)} \\ &= \frac{1}{4\pi}\left(\frac{\zeta + z}{\zeta - z} + \frac{\overline{\zeta} + \overline{z}}{\overline{\zeta} - \overline{z}}\right) \qquad (3.6.6) \\ &= \frac{1}{2\pi}\,\mathfrak{Re}\left(\frac{\zeta + z}{\zeta - z}\right). \end{aligned}$$

Thus, (3.6.5) is equivalent to

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\zeta)\,\mathfrak{Re}\left(\frac{\zeta + z}{\zeta - z}\right)\mathrm{d}t.$$

Since $\mathrm{d}\zeta = \mathrm{i}\zeta\,\mathrm{d}t$, $\mathrm{d}t = \frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}$, and

$$u(z) = \frac{1}{2\pi\mathrm{i}}\oint_{\partial D(0,R)} \frac{u(\zeta)}{\zeta}\,\mathfrak{Re}\left(\frac{\zeta + z}{\zeta - z}\right)\mathrm{d}\zeta = \mathfrak{Re}\left(\frac{1}{2\pi\mathrm{i}}\oint_{\partial D(0,R)} \frac{u(\zeta)}{\zeta}\frac{\zeta + z}{\zeta - z}\,\mathrm{d}\zeta\right),$$

where $z \in D(0, R)$. Since $R > 0$ and $\zeta - z \neq 0$ for all $\zeta \in \partial D(0, R)$ and $z \in D(0, R)$, the function

$$F(z) = \frac{1}{2\pi\mathrm{i}}\oint_{\partial D(0,R)} \frac{u(\zeta)}{\zeta}\frac{\zeta + z}{\zeta - z}\,\mathrm{d}\zeta$$

is holomorphic on $D(0, R)$: for each fixed $\zeta \in \partial D(0, R)$, the integrand is holomorphic in z, and on compact subsets of $D(0, R)$ we may differentiate under the integral sign. Therefore, $u(z)$ is the real part of a holomorphic function

$$f(z) = \frac{1}{2\pi\mathrm{i}}\oint_{\partial D(0,R)} \frac{u(\zeta)}{\zeta}\frac{\zeta + z}{\zeta - z}\,\mathrm{d}\zeta + \mathrm{i}c,$$

where $c \in \mathbb{R}$. Since $c \in \mathbb{R}$ is holomorphic, by Proposition 2.2.1, c is constant. For $f(z) = u(z) + \mathrm{i}v(z)$,

$$v(z) = c + \frac{1}{2\pi}\int_0^{2\pi} u(\zeta)\,\mathfrak{Im}\left(\frac{\zeta + z}{\zeta - z}\right)\mathrm{d}t. \qquad (3.6.7)$$

Letting $z = 0$, the integral vanishes, and we obtain $c = v(0) = \mathfrak{Im}(f(0))$.

Define the *Schwarz kernel* to be

$$S(\zeta, z) = \frac{\zeta + z}{2\pi\mathrm{i}(\zeta - z)\zeta}.$$

Then for a holomorphic function f on $D(0, R)$ that is continuous on $\overline{D(0, R)}$, we obtain the *Schwarz Integral Formula*:

$$f(z) = \oint_{\partial D(0,R)} \mathfrak{Re}(f(\zeta))S(\zeta, z)\,\mathrm{d}\zeta + \mathrm{i}\,\mathfrak{Im}(f(0)). \tag{3.6.8}$$

The significance of this alternative formula implies that a holomorphic function can be recovered from the real part on the boundary of a disk and the imaginary part at a single point.

From (3.6.7), we can rewrite

$$\begin{aligned} \mathfrak{Im}\left(\frac{\zeta + z}{\zeta - z}\right) &= \mathfrak{Im}\left(1 + \frac{2z}{\zeta - z}\right) \\ &= \mathfrak{Im}\left(\frac{2z\left(\overline{\zeta} - \overline{z}\right)}{|\zeta - z|^2}\right) \\ &= \frac{2\,\mathfrak{Im}\left(z\overline{\zeta}\right)}{|\zeta - z|^2}. \end{aligned} \tag{3.6.9}$$

Let

$$Q(\zeta, z) = \frac{\mathfrak{Im}\left(z\overline{\zeta}\right)}{\pi|\zeta - z|^2},$$

which is known as the *conjugate Poisson kernel.* Then (3.6.7) yields yet another integral representation of harmonic functions:

$$v(z) = v(0) + \int_0^{2\pi} u(\zeta)Q(\zeta, z)\,\mathrm{d}t.$$

where $\zeta = R\mathrm{e}^{\mathrm{i}t}$. Two harmonic functions are said to be *conjugate* if they are the real and imaginary parts of a holomorphic function. As seen above, on open disks, any harmonic function will admit a unique conjugate, up to an additive constant $v(0)$. For a harmonic function u, we can construct its harmonic conjugate from (3.6.9).

The Poisson kernel is important in many branches of mathematics. We will introduce two of the important uses below.

3.6.1 Solution to the Dirichlet Problem on a Disk

A fundamental problem in the theory of partial differential equations is to find a function u that is continuous on the closed disk $\overline{D(0, R)}$, harmonic on the open disk $D(0, R)$, and identically equal to a given boundary function on

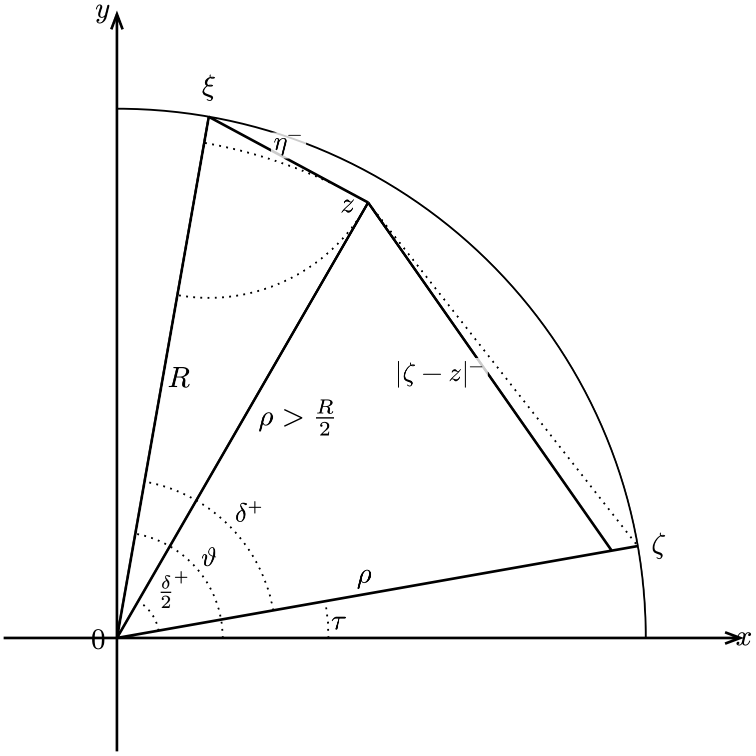

Figure 5: ζ, ξ, and z when $|\vartheta - \tau| > \delta$, with distances marked. The use of $+$ and $-$ denote a value more or less (respectively) than the preceding value.

$\partial D(0, R)$. This is known as the *Dirichlet problem* (for Laplace's equation) on a disk.

Theorem 3.6.1.1: For a continuous function $\varphi \in C^0(\partial D(0, R))$, the unique real-valued solution $u \in C^0\left(\overline{D(0, R)}\right)$ that solves

$$\begin{aligned} \nabla^2 u(z) &= 0 \quad \forall z \in D(0, R), \\ u(z) &= \varphi(z) \quad \forall z \in \partial D(0, R) \end{aligned}$$

is given by the Poisson integral formula:

$$u(z) = \int_0^{2\pi} \varphi(\zeta) P(\zeta, z) \,\mathrm{d}\tau, \tag{3.6.10}$$

where $\zeta = R\mathrm{e}^{\mathrm{i}\tau}$.

Proof: Since

$$P(\zeta, z) = \frac{1}{4\pi}\left(\frac{\zeta + z}{\zeta - z} + \frac{\overline{\zeta} + \overline{z}}{\overline{\zeta} - \overline{z}}\right),$$

from (3.6.7), we have that $\nabla_z^2 P(\zeta, z) = 4\frac{\partial^2 P(\zeta,z)}{\partial z \partial \overline{z}} = 0$ (since each term is independent of either z or $\overline{z}$). Moreover, by Theorem 1.2.6, (3.6.10) gives that

$$\nabla^2 u(z) = \nabla^2 \int_0^{2\pi} \varphi(\zeta) P(\zeta, z) \, \mathrm{d}\tau = \int_0^{2\pi} \nabla^2 [\varphi(\zeta) P(\zeta, z)] \, \mathrm{d}\tau = 0.$$

Our goal is to show that for fixed $\xi = R\mathrm{e}^{\mathrm{i}\vartheta} \in \partial D(0, R)$,

$$\lim_{\substack{z \to \xi \\ z \in D(0,R)}} u(z) = \varphi(\xi). \tag{3.6.11}$$

Let $\frac{R}{2} < \rho < R$ and $z = \rho \mathrm{e}^{\mathrm{i}\theta}$. Then with $\zeta = R\mathrm{e}^{\mathrm{i}\tau}$,

$$|\varphi(\xi) - u(z)| = \left|\varphi\left(R\mathrm{e}^{\mathrm{i}\vartheta}\right) - u\left(\rho \mathrm{e}^{\mathrm{i}\theta}\right)\right| = \left|\varphi\left(R\mathrm{e}^{\mathrm{i}\vartheta}\right) - \int_0^{2\pi} P(\zeta, z)\varphi(\zeta) \, \mathrm{d}\tau\right|.$$

For a constant harmonic function identically equal to 1, we get $\int_0^{2\pi} P(\zeta, z) \, \mathrm{d}\tau = 1$ from (3.6.5). Hence,

$$|\varphi(\xi) - u(z)| = \left|\int_0^{2\pi} P(\zeta, z)\left(\varphi\left(R\mathrm{e}^{\mathrm{i}\vartheta}\right) - \varphi(\zeta)\right) \mathrm{d}\tau\right|.$$

By the continuity of φ, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall |\vartheta - \tau| < \delta < \frac{\pi}{2}$, we have that $\left|\varphi\left(R\mathrm{e}^{\mathrm{i}\vartheta}\right) - \varphi(\zeta)\right| < \varepsilon$. Therefore,

$$\begin{aligned} |\varphi(\xi) - u(z)| &= \left|\left(\int_{|\vartheta - \tau| < \delta} + \int_{|\vartheta - \tau| > \delta}\right) P(\zeta, z)\left(\varphi\left(R\mathrm{e}^{\mathrm{i}\vartheta}\right) - \varphi(\zeta)\right) \mathrm{d}\tau\right| \\ &= |I_1 + I_2| \le |I_1| + |I_2|. \end{aligned}$$

Since the Poisson kernel is non-negative,

$$|I_1| < \int_{|\vartheta - \tau| < \delta} \varepsilon P(\zeta, z) \, \mathrm{d}\tau < \varepsilon \int_0^{2\pi} P(\zeta, z) \, \mathrm{d}\tau = \varepsilon.$$

By continuity of φ on the compact set $\partial D(0, R)$, by Theorem 1.2.15, it is bounded and $M = \sup_{|\zeta| = R} |\varphi(\zeta)|$ is finite. The Poisson kernel can be rewritten as

$$P(\zeta, z) = \frac{R^2 - \rho^2}{2\pi |\zeta - z|^2},$$

where $\zeta = R\mathrm{e}^{\mathrm{i}\tau}$ and $z = \rho\mathrm{e}^{\mathrm{i}\theta}$, with $|\vartheta - \tau| > \delta$. Then $\exists \eta > 0$ such that $\forall z$ with $|\xi - z| < \eta$ (small enough so that $|\vartheta - \theta| < \frac{\delta}{2}$),

$$|\theta - \tau| > \frac{\delta}{2} \tag{3.6.12}$$

and

$$\left(\rho > \frac{R}{2}\right) \quad \text{and} \quad \eta \leq \frac{R}{2} \tag{3.6.13}$$

as in Figure 5. Then,

$$|\zeta - z|^2 > 4\rho^2 \sin\left(\frac{\delta}{4}\right)^2 > \frac{1}{2}R^2\left(1 - \cos\left(\frac{\delta}{2}\right)\right).$$

We aim to prove that $|I_2| < \varepsilon$. Since $\left|\varphi\left(R\mathrm{e}^{\mathrm{i}\vartheta}\right) - \varphi(\zeta)\right| < 2M$, the condition is satisfied if

$$\int_{|\vartheta - \tau| > \delta} \frac{R^2 - \rho^2}{\pi R^2\left(1 - \cos\left(\frac{\delta}{2}\right)\right)} \,\mathrm{d}\tau < 2\frac{R^2 - \rho^2}{R^2\left(1 - \cos\left(\frac{\delta}{2}\right)\right)} < \frac{\varepsilon}{2M},$$

and from rearrangement, we can tighten the constraint with:

$$R^2 - \rho^2 < \frac{\varepsilon}{4M}R^2\left(1 - \cos\left(\frac{\delta}{2}\right)\right),$$

which follows in particular from

$$R - \rho < \frac{\varepsilon}{8M}R\left(1 - \cos\left(\frac{\delta}{2}\right)\right). \tag{3.6.14}$$

From Figure 5, it is evident that $R - \rho < |\xi - z| < \eta$. For (3.6.12) to be true, we previously had that $|\vartheta - \theta| < \frac{\delta}{2}$. In other words

$$|\xi - z|^2 < R^2 + \rho^2 - 2R\rho\cos\left(\frac{\delta}{2}\right).$$

Obviously, this is satisfied if $|\xi - z|^2 < \frac{R^2}{2}\left(1 - \cos\left(\frac{\delta}{2}\right)\right) < 2\rho^2\left(1 - \cos\left(\frac{\delta}{2}\right)\right)$. This can be rearranged into

$$|\xi - z| < R\sqrt{\frac{1 - \cos\left(\frac{\delta}{2}\right)}{2}} = R\sin\left(\frac{\delta}{4}\right).$$

Therefore, we can choose

$$\eta = \min\left(\frac{\varepsilon}{8M} R\left(1 - \cos\left(\frac{\delta}{2}\right)\right), R\sin\left(\frac{\delta}{4}\right), \frac{R}{2},\right) > 0,$$

under which (3.6.12), (3.6.13), and (3.6.14) are satisfied.

Hence, $\forall \varepsilon > 0$, $\exists \eta > 0$ such that $\forall z$ with $0 < |\xi - z| < \eta$, we have $|\varphi(\xi) - u(z)| < 2\varepsilon$. Then (3.6.11) follows.

We will now show that $u(z)$ is unique. Assume that $v \neq u$ on $\overline{D(0,R)}$ also solves the problem. Then $u - v$ is harmonic and vanishes on $\partial D(0,R)$. By the Poisson Integral Formula ((3.6.5)),

$$u(z) - v(z) = \int_0^{2\pi} P(\zeta, z)(u(\zeta) - v(\zeta))\,\mathrm{d}\tau = 0$$

for all $z \in D(0,R)$. Hence $u \equiv v$, a contradiction. $\square$

3.6.2 In Harmonic Analysis

Consider $R = 1$, $\zeta = \mathrm{e}^{\mathrm{i}\tau}$, and $z = \rho\mathrm{e}^{\mathrm{i}\theta}$ in (3.6.5):

$$\begin{aligned} u(z) &= \frac{1}{2\pi}\int_0^{2\pi} u(\zeta)\frac{1 - |z|^2}{|\zeta - z|^2}\,\mathrm{d}\tau \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-\rho^2)u(\mathrm{e}^{\mathrm{i}\tau})\,\mathrm{d}\tau}{(\mathrm{e}^{\mathrm{i}\tau} - \rho\mathrm{e}^{\mathrm{i}\theta})(\mathrm{e}^{-\mathrm{i}\tau} - \rho\mathrm{e}^{-\mathrm{i}\theta})} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-\rho^2)u(\mathrm{e}^{\mathrm{i}\tau})\,\mathrm{d}\tau}{1 + \rho^2 - 2\rho\cos(\theta - \tau)}. \end{aligned} \tag{3.6.15}$$

Since $u(z)$ is continuous on $\partial\mathbb{D}$ and $u\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ is periodic with period 2π, it admits a Fourier series representation with coefficients

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} u(\mathrm{e}^{\mathrm{i}\tau})\mathrm{e}^{-\mathrm{i}n\tau}\,\mathrm{d}\tau, \tag{3.6.16}$$

so that the corresponding Fourier series is

$$\sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\theta}.$$

This series may diverge. Observe that continuity of u on the compact set $\partial\mathbb{D}$ implies uniform boundedness: $\exists M > 0$ such that $\left|u\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| \leq M$ for all θ (Theorem 1.2.13). Consequently, $|a_n| \leq M$. Introducing factors $\rho^{|n|}$ with $|\rho| < 1$ yields a convergent series:

$$\sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\theta}\rho^{|n|}, \qquad \left|\sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\theta}\rho^{|n|}\right| \le \sum_{n=-\infty}^{\infty} |a_n|\rho^{|n|} \le M\frac{1+|\rho|}{1-|\rho|}.$$

Substituting the coefficients gives

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\theta}\rho^{|n|} &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi}\int_0^{2\pi} u(\mathrm{e}^{\mathrm{i}\tau})\mathrm{e}^{-\mathrm{i}n\tau}\,\mathrm{d}\tau\right)\mathrm{e}^{\mathrm{i}n\theta}\rho^{|n|} \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_0^{2\pi} \rho^{|n|}u(\mathrm{e}^{\mathrm{i}\tau})\mathrm{e}^{\mathrm{i}n(\theta-\tau)}\,\mathrm{d}\tau.
\end{aligned}$$

By Theorem 2.3.2 and Theorem 2.3.6,

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_0^{2\pi} \rho^{|n|}u(\mathrm{e}^{\mathrm{i}\tau})\mathrm{e}^{\mathrm{i}n(\theta-\tau)}\,\mathrm{d}\tau = \frac{1}{2\pi}\int_0^{2\pi} u(\mathrm{e}^{\mathrm{i}\tau})\sum_{n=-\infty}^{\infty}\rho^{|n|}\mathrm{e}^{\mathrm{i}n(\theta-\tau)}\,\mathrm{d}\tau. \tag{3.6.17}$$

The summation simplifies as follows:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\rho^{|n|}\mathrm{e}^{\mathrm{i}n(\theta-\tau)} &= \sum_{n=0}^{\infty}\rho^{n}\mathrm{e}^{\mathrm{i}n(\theta-\tau)} + \sum_{n=1}^{\infty}\rho^{n}\mathrm{e}^{-\mathrm{i}n(\theta-\tau)} \\
&= 1 + 2\sum_{n=1}^{\infty}\rho^{n}\cos(n(\theta-\tau)) \\
&= 1 + 2\,\mathfrak{Re}\left(\sum_{n=1}^{\infty}\rho^{n}\mathrm{e}^{\mathrm{i}n(\theta-\tau)}\right) \\
&= 1 + 2\,\mathfrak{Re}\left(\frac{\rho\mathrm{e}^{\mathrm{i}(\theta-\tau)}}{1-\rho\mathrm{e}^{\mathrm{i}(\theta-\tau)}}\right) \\
&= \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\tau)}.
\end{aligned}$$

Substituting into (3.6.17) yields

$$\sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\theta}\rho^{|n|} = \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-\rho^2)u(\mathrm{e}^{\mathrm{i}\tau})}{1+\rho^2-2\rho\cos(\theta-\tau)}\,\mathrm{d}\tau = u(\rho\mathrm{e}^{\mathrm{i}\theta}).$$

Furthermore, by the proof of Theorem 3.6.1.1 (specifically (3.6.11)),

$$\lim_{\rho\to1^-}\sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\theta}\rho^{|n|} = u(\mathrm{e}^{\mathrm{i}\theta}).$$

Thus, for any continuous function u on $\partial\mathbb{D}$, its Fourier series is *Abel summable* to u.

We now establish that real-valued continuous functions satisfying the mean-value property are harmonic.

Theorem 3.6.2.1: Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{R}$ continuous. Suppose for every $z_0 \in U$, there exists $\lambda > 0$ with $\overline{D(z_0, \lambda)} \subseteq U$ such that for all $0 < \varepsilon \leq \lambda$,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \varepsilon \mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t.$$

Then f is harmonic on U.

Proof: Fix $z_0 \in U$ arbitrarily and choose $\lambda > 0$ such that $\overline{D(z_0, \lambda)} \subseteq U$. Because $f \in C^0(\partial D(z_0, \lambda))$, Theorem 3.6.1.1 guarantees the existence of a unique harmonic function u on $D(z_0, \lambda)$ satisfying

$$u(z) = \int_0^{2\pi} f(\zeta) P(\zeta, z)\,\mathrm{d}\tau,$$

with $u = f$ on $\partial D(z_0, \lambda)$. Define $\psi = f - u$ on $\overline{D(z_0, \lambda)}$. Then ψ is continuous, satisfies the mean-value property, and vanishes on $\partial D(z_0, \lambda)$. By the proof of Theorem 3.4.1, which relies solely on the mean-value property, $\psi \equiv 0$ on $\overline{D(z_0, \lambda)}$. Thus, $f \equiv u$ on $\overline{D(z_0, \lambda)}$, implying f is harmonic at z_0. The arbitrariness of z_0 establishes harmonicity on U. □

4 The Theory of Weierstrass

While Weierstrass' contributions in complex analysis are mainly characterized by his discoveries on uniform convergence, he also characterized entire and *meromorphic functions* and a unique representation of entire functions, as well as his contributions toward the study of *essential singularities*.

To classify the behavior of non-removable singularities, mathematicians generalized Taylor series to *Laurent series*.

4.1 Laurent Series

The Laurent series generalizes the Taylor series to holomorphic functions with isolated singularities. While Taylor series are valid within a disk centered at a point of holomorphy, Laurent series apply to annular regions surrounding a singularity, making them essential for studying functions near non-removable singularities (refer to Theorem 3.2.6).

We now introduce a fundamental result in complex analysis due to Weierstrass, which formalizes the conditions under which the limit of a sequence of holomorphic functions is itself holomorphic. This theorem not only guarantees the holomorphy of the limit function but also the uniform convergence of its derivatives (its statement was used in the proof of Theorem 3.3.5).

Theorem 4.1.1 *(Weierstrass)*: Let $\{f_n(z)\}_{n\in\mathbb{N}}$ be a sequence of holomorphic functions on an open region $U \subseteq \mathbb{C}$ that converges uniformly to $f(z)$ on every compact subset of U. Then $f(z)$ is holomorphic on U, and $\forall k \in \mathbb{N}$, the sequence $\{f_n^k(z)\}_{n\in\mathbb{N}}$ uniformly converges to $f^{(k)}(z)$ on all compact subsets of U.

Proof: By Morera's Theorem (Theorem 3.2.4) and the uniform convergence of $\{f_n(z)\}$, the holomorphy of $f(z)$ follows (refer to (4.1.2) and preceding explanations).

Following the same logic, by Corollary 3.2.5.1, $\forall k \in \mathbb{N}$ and for all compact $K \subset U$ and open $V \supset K$ relatively compact in U there exists a finite constant $c_k > 0$ such that

$$\lim_{n\to\infty} \sup_{z\in K} \left| f_n^{(k)}(z) - f^{(k)}(z) \right| \leq c_k \lim_{n\to\infty} \sup_{z\in V} |f_n(z) - f(z)|.$$

Since $\{f_n(z)\}$ is uniformly convergent, the limit on the right-hand side vanishes. Then,

$$\lim_{n\to\infty} \sup_{z\in K} \left| f_n^{(k)}(z) - f^{(k)}(z) \right| = 0,$$

and therefore $\left\{f_n^{(k)}(z)\right\}_{n\in\mathbb{N}}$ uniformly converges on all compact subsets of U. □

The condition of uniform convergence on every compact subset can also be significantly loosened, by the fact demonstrated below:

Proposition 4.1.1: Let $U \subseteq \mathbb{C}$ be an open bounded region, and let $\{f_n(z)\}$ be holomorphic on U. Let $K \subset U$ be compact. If $f_n(z) \rightrightarrows f(z)$ on ∂K, then $f_n(z) \rightrightarrows f(z)$ on K.

Proof: By the converse statement of the Cauchy Criterion (Theorem 2.3.1), $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n, m > N$,

$$\sup_{z\in\partial K} |f_n(z) - f_m(z)| < \varepsilon.$$

By the Maximum Modulus Principle (Theorem 3.4.1) on $f_n - f_m$,

$$\sup_{z\in\partial K} |f_n(z) - f_m(z)| = \sup_{z\in K} |f_n(z) - f_m(z)| < \varepsilon.$$

It follows that $f_n(z) \rightrightarrows f(z)$ on K by Theorem 2.3.1. □

Remark: From the above result, the uniform convergence on every compact subset in Theorem 4.1.1 can therefore be loosened to the uniform convergence on every simple closed curve.

We will now study Laurent series. Let $a \in \mathbb{C}$ and $\{c_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ be constants. A series in the form of

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \tag{4.1.1}$$

is a Laurent series at the point a. The series can be separated into a power series with non-negative exponents,

$$\varphi(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \tag{4.1.2}$$

and a power series with negative exponents,

$$\psi(z) = \sum_{n=1}^{\infty} c_{-n} (z-a)^{-n}. \tag{4.1.3}$$

(4.1.1) is said to be convergent at $z = z_0$ if the two power series are both convergent. Let the convergence radius of (4.1.2) be

$$R = \frac{1}{\limsup_{n\to\infty}} \sqrt[n]{|c_n|}$$

by the Cauchy–Hadamard Theorem (Theorem 2.3.4). It follows that φ is holomorphic on $D(a, R)$. Let $\zeta = (z-a)^{-1}$. Then (4.1.3) becomes

$$\sum_{n=1}^{\infty} c_{-n} \zeta^n.$$

This series converges when

$$|\zeta| < \frac{1}{\limsup_{n\to\infty}} \sqrt[n]{|c_{-n}|} = \lambda.$$

Let $r = \frac{1}{\lambda}$. Then $\psi(z)$ converges when

$$|z-a| > \limsup_{n\to\infty} \sqrt[n]{|c_{-n}|},$$

or when $z \in \mathbb{C} \setminus \overline{D(a,r)}$.

If $R > r$, then f is convergent on the annulus $D(a,R) \setminus \overline{D(a,r)}$ and divergent on $\left(\mathbb{C} \setminus \overline{D(a,R)}\right) \cup D(a,r)$. If $r = R$, the series diverges possibly everywhere but on $\partial D(a, r)$. Similar to power series with positive exponents, the convergence on the boundary varies. For example,

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{z^n}{n^2},$$

where $R = r = 1$, converges (absolutely) on $\partial\mathbb{D}$, whereas

$$\sum_{n=-\infty}^{\infty} z^n$$

diverges on all of $\partial\mathbb{D}$, while

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{z^n}{n}$$

converges (conditionally) on all of $\partial\mathbb{D} \setminus \{1\}$ and diverges at $z = 1$. If $r > R$, then the series is divergent on all of $\mathbb{C}$. The region $D(a, R) \setminus \overline{D(a, r)}$ is known as the *annulus of convergence*. $f(z)$ in (4.1.1) is holomorphic over this annulus. The series $\varphi(z)$ is known as the *holomorphic part* of $f(z)$, and $\psi(z)$ is known as the *principal part* of the Laurent series. The properties of the convergence disk in Abel's Theorem (Theorem 2.3.3) can be generalized to Laurent series. In other words, f is absolutely convergent on the annulus and is uniformly convergent on every compact subset of it.

Theorem 4.1.2: Let $V = \{z \in \mathbb{C} : r < |z - a| < R\}$ for some $0 \leq r < R \leq \infty$. Let f be holomorphic on V. Then f has the unique *Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n, \quad c_n = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - a)^{n+1}}, \quad z \in V, \tag{4.1.4}$$

for any simple closed curve $\gamma \subset V$ enclosing a. Moreover, the series converges absolutely on V and uniformly on all compact subsets of V.

Proof: By the openness of V, there exist two circles $\gamma_1 \subset V$ with radius r' and $\gamma_2 \subset V$ with radius R' centered at a such that γ encloses γ_1 and γ_2 encloses γ both without intersection. Let $W = \{z \in V : r' < |z - a| < R'\}$ and let $z \in W$ be arbitrary. By the Cauchy–Goursat Formula (Theorem 3.1.8),

$$f(z) = \frac{1}{2\pi \mathrm{i}} \left(\oint_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \oint_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \right).$$

For all $\zeta \in \gamma_1$ (or $|\zeta - a| = r'$), $|\zeta - a| < |z - a|$ and therefore, $\frac{|\zeta - a|}{|z - a|} < 1$. It follows that

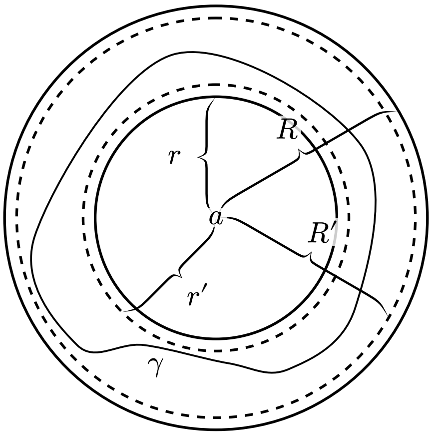

Figure 6: The annulus V, with γ_1, γ_2, and γ.

$$\frac{1}{\zeta - z} = -\frac{1}{(z-a)\left(1 - \frac{\zeta - a}{z-a}\right)} = -\sum_{n=0}^{\infty} \frac{(\zeta - a)^n}{(z-a)^{n+1}} \tag{4.1.5}$$

is uniformly convergent with respect to ζ. Similarly, for all $\zeta \in \gamma_2$,

$$|\zeta - a| > |z - a| \iff \frac{|z-a|}{|\zeta - a|} < 1,$$

and it follows that

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a)\left(1 - \frac{z-a}{\zeta - a}\right)} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^{n+1}} \tag{4.1.6}$$

is uniformly convergent with respect to ζ. By the boundedness of f on γ_1 and γ_2 from holomorphy on a compact set, the uniform convergence from the Weierstrass M-Test (Theorem 2.3.2), gives that

$$f(z) = \frac{1}{2\pi\mathrm{i}}\left(\sum_{n=0}^{\infty} \oint_{\gamma_2} \frac{(z-a)^n}{(\zeta - a)^{n+1}} f(\zeta)\,\mathrm{d}\zeta + \sum_{n=1}^{\infty} \oint_{\gamma_1} \frac{(\zeta - a)^{n-1}}{(z-a)^n} f(\zeta)\,\mathrm{d}\zeta\right) \tag{4.1.7}$$

By the Cauchy–Goursat Theorem (Theorem 3.1.7), for a given n,

$$\int_{\gamma_2^+ \cup \gamma^-} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - a)^n} = 0 \quad \text{and} \quad \int_{\gamma^+ \cup \gamma_1^-} f(\zeta)(\zeta - a)^n\,\mathrm{d}\zeta = 0.$$

In other words, the integrals in (4.1.7) are the same as on γ. Hence, we obtain the absolutely convergent expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{\infty} c_{-n} (z-a)^{-n} = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

which converges uniformly on compact sets of V. The constants $\{c_n\}_{n \in \mathbb{Z}}$ are also unique in the expansion. For the sake of contradiction, assume there exists another set of constants $\{c_n'\}_{n \in \mathbb{Z}}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n' (z-a)^n, \tag{4.1.8}$$

where $z \in V$ and the series is uniformly convergent on γ. Let $m \in \mathbb{Z}$ be arbitrary. By Cauchy–Goursat (Theorem 3.2.1),

$$\oint_\gamma (z-a)^k \, \mathrm{d}z = \begin{cases} 0 & \text{if } k \geq 0 \\ 2\pi \mathrm{i} \frac{\mathrm{d}^{-k-1}}{\mathrm{d}z^{-k-1}}(1) & \text{if } k \leq -1 \end{cases} = \begin{cases} 0 & \text{if } k \neq -1 \\ 2\pi \mathrm{i} & \text{if } k = -1. \end{cases}$$

Multiplying (4.1.8) by $(z-a)^{-m-1}$ and from integrating over γ, we get that

$$\oint_\gamma \frac{f(z)\, \mathrm{d}z}{(z-a)^{m+1}} = \oint_\gamma \sum_{n=-\infty}^{\infty} c_n' (z-a)^{n-m-1} \, \mathrm{d}z,$$

implying that

$$2\pi \mathrm{i} c_m = \sum_{n=-\infty}^{\infty} c_n' \oint_\gamma (z-a)^{n-m-1} \, \mathrm{d}z = 2\pi \mathrm{i} c_m',$$

which is a contradiction, implying uniqueness. ☐

Remark: Unlike Taylor series, Laurent series are not necessarily unique up to the point of expansion. Depending on the chosen annulus, the expansion may differ.

4.2 Isolated Singularities

An *isolated singularity* of a complex function is a point $a \in \mathbb{C}$ where a function f is holomorphic on some open punctured neighborhood of a (namely, for some $r > 0$, the punctured disk $D^*(a, r)$), but not necessarily defined or holomorphic at a itself. The nature of this isolated singularity is characterized by the principal part $\psi(z)$ (let $\varphi(z)$ be the holomorphic part) of the Laurent series of f at the point a. Specifically, we can analyze the behavior of $f(z)$ as $z \to a$.

1. If $\lim_{z \to a} f(z)$ exists and is finite, then $z = a$ is a removable singularity and can be analytically continued to $D(a, r)$ by Theorem 3.2.6. Consequently, $f(z)$ has a convergent Taylor expansion and the principal part of its Laurent expansion vanishes, and $f(z) = \varphi(z)$.

2 If $\lim_{z\to a} f(z) = \infty$, then $z = a$ is a *pole* of f (from the stereographic projection and the Riemann sphere, the ∞ is a single point in $\hat{\mathbb{C}}$, and approaching ∞ does not distinguish between different directions, unlike the use of $+\infty$ and $-\infty$).

Theorem 4.2.1: The condition $\lim_{z\to a} f(z) = \infty$ is equivalent to there being a finite number of nonzero c_{-n}'s, where $n \in \mathbb{N}$.

In other words the principal part of f is equal to

$$\psi(z) = \frac{c_{-1}}{z-a} + \cdots + \frac{c_{-m}}{(z-a)^m}, \quad c_{-m} \neq 0$$

for some $m \in \mathbb{N}$. Therefore,

$$f(z) = \varphi(z) + \psi(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n = \frac{g(z)}{(z-a)^m}$$

on the punctured disk $D^*(a,r)$, where

$$g(z) = \sum_{n=0}^{\infty} c_{n-m} (z-a)^n$$

is holomorphic on $D(a,r)$ and does not attain a zero at $z = a$. Then $f(z)$ has a pole at $z = a$ with order m. If $m = 1$, the pole is also called a *simple pole*.

Proof: Obviously, under the assumption of a finite, nonempty number of nonzero terms in the principal part of the Laurent expansion coefficients, $\lim_{z\to a} f(z) = \infty$. Now we will prove the converse. Let

$$g(z) = \frac{1}{f(z)}.$$

Then $\lim_{z\to a} g(z) = 0$. There exists a $\delta > 0$ such that f is nonzero on $D^*(a,\delta)$. Then $g(z)$ is holomorphic on $D^*(a,\delta)$ and has a removable singularity at $z = a$. By Theorem 3.2.6, g can be analytically continued to $D(a,\delta)$. Let the multiplicity of the zero at $z = a$ be m. Then

$$g(z) = \varphi(z)(z-a)^m,$$

where $\varphi(z)$ is holomorphic and nonzero at $z = a$. Then there exists a $\delta' > 0$ such that φ is nonzero on $D(a,\delta')$. It follows that $\frac{1}{\varphi}$ is holomorphic and nonzero on $D(a,\delta')$. We can then write its Taylor expansion as

$$\frac{1}{\varphi(z)} = c_{-m} + c_{1-m}(z-a) + \cdots,$$

where $c_{-m} \neq 0$. It follows that

$$f(z) = \frac{1}{g(z)} = \frac{(z-a)^{-m}}{\varphi(z)} = c_{-m}(z-a)^{-m} + \cdots + c_0 + \cdots.$$

By the uniqueness of the Laurent series, the conclusion follows. □

3. If $\lim_{z \to a} f(z)$ is nonexistent, then a is known as an *essential singularity.*

   Example 4.2.1: The function $\mathrm{e}^{\frac{1}{z}}$ has an essential singularity at $z = 0$.

   Proof: Observe that $\lim_{x \to 0^+} \mathrm{e}^{\frac{1}{x}} = \infty$. Similarly, $\lim_{x \to 0^-} \mathrm{e}^{\frac{1}{x}} = 0$, and for $z = \mathrm{i}y$ with $y \to 0^+$,

   $$\mathrm{e}^{\frac{1}{z}} = \mathrm{e}^{-\frac{\mathrm{i}}{y}},$$

   which is divergent. Therefore, the limit does not exist. □

   The implication on its Laurent expansion at a is:

   Theorem 4.2.2: The necessary and sufficient condition for $\lim_{z \to a} f(z)$ to not exist is that infinitely many of c_{-n} (where $n \in \mathbb{N}$) are nonzero.

   This follows by elimination from the established trichotomy; if the limit as $z \to a$ does not exist, then the singularity is neither removable nor a pole (results from Part 1 and Part 2). Similar logic can be applied to the coefficients of the Laurent expansion.

   Indeed, in Example 4.2.1, the Laurent expansion is equal to:

   $$\mathrm{e}^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!},$$

   which has infinitely many nonzero coefficients of negative powers.

A function with an essential singularity exhibits striking behavior. We will first introduce the following famous result.

Theorem 4.2.3 *(Casorati–Sokhotski–Weierstrass)*: Let $a \in \mathbb{C}$ and $U \subseteq \mathbb{C}$ be an open region. Suppose $f : U \setminus \{a\} \to \mathbb{C}$ is holomorphic with an essential singularity at a. Then the set of values that f attains on any open punctured neighborhood of a is dense. In other words, $\forall \varepsilon, \delta > 0, \forall w \in \mathbb{C}, \exists z \in D^*(a, \delta)$ such that $|f(z) - w| < \varepsilon$.

Proof: Assume for the sake of contradiction that $\exists \varepsilon, \delta > 0$, and $\exists w \in \mathbb{C}$ such that $\forall z \in D^*(a, \delta)$, $|f(z) - w| > \varepsilon$. Define the auxiliary function

$$g(z) = \frac{f(z) - w}{z - a},$$

which is holomorphic and non-vanishing on the punctured neighborhood of a. Since as $z \to a$, $g(z) \to \infty$, it follows that $g(z)$ has a pole at a. Let the order of the pole be $m \in \mathbb{N}$. By Theorem 4.2.1, $g(z)$ has the Laurent expansion

$$\frac{c_{-m}}{(z-a)^m} + \cdots + c_0 + c_1(z-a) + \cdots$$

for some $m \in \mathbb{N}$. It follows that

$$f(z) = \frac{c_{-m}}{(z-a)^{m-1}} + \cdots + c_{-1} + w + c_0(z-a) + \cdots.$$

If $m = 1$, then f has a removable singularity at a. If $m \geq 2$, then f has a pole at a. Hence, we have a contradiction. $\square$

An analogous proof yields the following result for entire functions.

Theorem 4.2.4: The set of values that a non-constant entire function f assumes is dense in $\mathbb{C}$.

Proof: For the sake of contradiction, assume there exists $w \in \mathbb{C}$ and $\varepsilon > 0$ such that $D(w, \varepsilon) \cap f(\mathbb{C}) = \varnothing$. Define

$$g(z) = \frac{1}{f(z) - w}.$$

It follows that $|g| \leq \frac{1}{\varepsilon}$ on $\mathbb{C}$. By Liouville's Theorem (Theorem 3.2.3), g is a constant function, and hence, f is also constant, which is a contradiction of the statement. $\square$

In Section 8, we will prove a profound generalization of the two results (@ thm:greatpicard and Theorem 8.3.4), which was first proved by Emile Picard in 1879.

4.2.1 At the ∞ Point

Given the one-point compactification of $\mathbb{C}$, $\hat{\mathbb{C}}$, we can now define and analyze the behavior of functions near the point at ∞. Similar to the classification of isolated singularities in $\mathbb{C}$, we can classify ∞ as a removable singularity, a pole, or an essential singularity of a holomorphic function.

Let $f : \mathbb{C} \setminus \overline{D(0,R)} \to \mathbb{C}$ be holomorphic for some $R > 0$. Then $z = \infty$ is an *isolated singularity* of f. To analyze the nature of the singularity, let $\zeta = \frac{1}{z}$. We define a new function $g(\zeta) = f\left(\frac{1}{\zeta}\right) = f(z)$, which is holomorphic on $D^*\left(0, \frac{1}{R}\right)$. Then at $\zeta = 0$, $g(\zeta)$ has the Laurent expansion of

$$g(\zeta) = \sum_{n=-\infty}^{\infty} c_{-n}\zeta^n = \sum_{n=0}^{\infty} c_{-n}\zeta^n + \sum_{n=1}^{\infty} c_n\zeta^{-n} = \varphi(\zeta) + \psi(\zeta),$$

where φ and ψ are the holomorphic and principal parts of g, respectively. Let $\tilde{\varphi}(z) = \varphi\left(\frac{1}{z}\right)$, $\tilde{\psi}(z) = \psi\left(\frac{1}{z}\right)$. At $z = 0$, f then has the Laurent expansion of

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_{-n} z^{-n} + \sum_{n=1}^{\infty} c_n z^n = \tilde{\varphi}(z) + \tilde{\psi}(z).$$

The classification of the singularity at ∞ is then reduced to the classification of the singularity of g at 0:

1 If $z = \infty$ is a removable singularity of $f(z)$, then $f(z)$ has the form of

$$f(z) = c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \frac{c_{-3}}{z^3} + \cdots.$$

2 If $z = \infty$ is a pole of $f(z)$ with degree $m \in \mathbb{N}$, then $f(z)$ can be written as

$$f(z) = c_m z^m + c_{m-1} z^{m-1} + \cdots + c_0 + \frac{c_{-1}}{z} + \cdots,$$

where $c_m \neq 0$.

3 If $z = \infty$ is an essential singularity of $f(z)$, then $f(z)$ can be expanded as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n,$$

where $\forall N \in \mathbb{N}, \exists n > N$ such that $c_n \neq 0$ (infinitely many coefficients of ψ or $\tilde{\psi}$ are nonzero).

Remark: Under stereographic projection from the point $(0, 0, 1)$ of the unit sphere S^2, a neighborhood of that point maps to a subset of the extended complex plane of the form $\hat{\mathbb{C}} \setminus K$, where K is a compact and connected subset of $\mathbb{C}$. Such sets are referred to as *neighborhoods of* ∞ in the Riemann sphere.

Example 4.2.1.1: The function $z \mapsto \frac{1}{z}$ has a removable singularity at $z = \infty$, the function $z \mapsto z^2$ has a pole at $z = \infty$, and $z \mapsto \mathrm{e}^z$ has an essential singularity at $z = \infty$.

4.3 Entireness and Meromorphy

We have previously defined the concept of an entire function in the chapter on complex differentiation. Let f be entire with the unique Taylor expansion $\sum_{n=0}^{\infty} c_n z^n$. Since $z = \infty$ is an isolated singularity, by the uniqueness of the Laurent expansion, the expansion at $z = 0$ has the same form as the expansion at $z = \infty$. We will now analyze the implications on the entire function f given an isolated singularity.

1 If the infinity point is a removable singularity, then $\lim_{z \to \infty} f(z)$ exists and is finite.

Proposition 4.3.1: If $f(z)$ is entire and has a removable singularity at $z = \infty$, then f is constant.

Proof: Let $z = \frac{1}{\zeta}$, and let $g(\zeta) = f\left(\frac{1}{\zeta}\right)$, which has a removable singularity at $\zeta = 0$. By Theorem 3.2.6, g can be analytically continued to all of $\mathbb{C}$, especially at $\zeta = 0$. Let $w = g(0)$. Then, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall \zeta \in D(0, \delta)$, $|g(\zeta) - w| < \varepsilon$. It follows that $\underline{\forall |z| > \frac{1}{\delta}}$, $|f(z)| < |w| + \varepsilon$, and is bounded. For the complement, $\forall z \in \overline{D\left(0, \frac{1}{\delta}\right)}$, $f(z)$ is continuous on a compact set, and by Theorem 1.2.13, is also bounded.

Then by Liouville's Theorem (Theorem 3.2.3), f is constant. □

2 If $f(z)$ has a pole at $z = \infty$ of order $m \in \mathbb{N}$, then f is a polynomial of degree m.

Proof: By the classification of a pole at ∞, f can be written as

$$f(z) = c_m z^m + c_{m-1} z^{m-1} + \cdots + c_0 + \frac{c_{-1}}{z} + \cdots.$$

Since $f(z)$ is entire, it is holomorphic at $z = 0$ and has a convergent Taylor expansion. By the uniqueness of Laurent expansions (Theorem 4.1.2), the two expansions are equivalent and therefore all terms with negative exponents vanish, and

$$f(z) = c_m z^m + c_{m-1} z^{m-1} + \cdots + c_0,$$

and since $c_m \neq 0$, the statement is confirmed. □

3 If $f(z)$ has an essential singularity at $z = \infty$, $f(z)$ is known as a *transcendental entire function.*

Example 4.3.1: The entire functions $\sin(z)$, $\cos(z)$, $\sinh(z)$, $\cosh(z)$, and $\exp(z)$ are transcendental.

Definition 4.3.1 *(Meromorphy)*: Let $U \subseteq \mathbb{C}$ be open, and let $\{a_n\}_{n \in \mathbb{N}} \subset U$ be a set of isolated points. Suppose $f : U \setminus \{a_n\}_{n \in \mathbb{N}} \to \mathbb{C}$ is holomorphic and has a pole at each of $z \in \{a_n\}$. Then f is *meromorphic* in U.

Similar to holomorphy, meromorphy on a compact set can be defined as meromorphy on a neighborhood of the set. In general, we imply for the set to be open unless stated otherwise. If the set is not implicitly specified, we assume meromorphy on $\mathbb{C}$.

All holomorphic functions are meromorphic functions (with poles on $\varnothing$). Consequently, entire functions are meromorphic on $\mathbb{C}$. All rational functions (including polynomials) are also meromorphic on $\mathbb{C}$. In the study of meromorphic functions with an isolated singularity at ∞, rational functions are of important interest.

Let $f(z)$ be rational, written as $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials. Let

$$\begin{aligned} p(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \\ q(z) &= b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0, \end{aligned}$$

where $a_n, b_m \neq 0$. Trivially, the poles of f are the zeros of q. Since

$$f(z) = \frac{z^n}{z^m} \cdot \frac{a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}}{b_m + \frac{b_{m-1}}{z} + \cdots + \frac{b_0}{z^m}},$$

we have

$$\lim_{z \to \infty} f(z) = \begin{cases} \frac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \\ \infty & \text{if } n > m. \end{cases}$$

Conversely, we have:

Theorem 4.3.1: If $f(z)$ is meromorphic on $\mathbb{C}$ and has a pole or removable singularity at $z = \infty$, then f is a rational function.

Proof: Since f is meromorphic on $\mathbb{C}$, its singularities are isolated poles. The assumption that f has either a pole or a removable singularity at ∞ implies that this singularity is also isolated. Thus, there exists some $R > 0$ such that f is holomorphic on the punctured neighborhood $\{z \in \mathbb{C} : R < |z| < \infty\}$ of ∞.

Consider the Laurent expansion of f at ∞, obtained by substituting $w = \frac{1}{z}$ and expanding around $w = 0$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where the series converges for sufficiently large $|z|$. If ∞ is a removable singularity, the coefficients $a_n = 0$ for all $n > 0$. If ∞ is a pole of order m, then $a_n = 0$ for all $n > m$, and $a_m \neq 0$. In either case, the principal part at ∞ is

$$\psi_\infty(z) = \sum_{n=1}^{m} a_n z^n,$$

which is a polynomial (identically zero if degree is 0).

Next, observe that f has only finitely many poles in the closed disk $\overline{D(0, R)} = \{z : |z| \leq R\}$. Suppose otherwise. Then the set of poles in $\overline{D(0, R)}$ would be infinite. By Bolzano–Weierstrass (Theorem 1.1.2), this set would have an accumulation point in $\overline{D(0, R)}$. At such an accumulation point, f would have a non-isolated singularity, a contradiction of the meromorphy of f on $\mathbb{C}$.

Let $z_1, ..., z_n$ denote these finitely many poles in $\overline{D(0, R)}$. For each $k = 1, ..., n$, the Laurent expansion of f at z_k has principal part

$$\psi_k(z) = \sum_{j=1}^{m_k} \frac{c_{k,-j}}{(z - z_k)^j},$$

where m_k is the order of the pole at z_k. Define the auxiliary function

$$\Phi(z) = f(z) - \psi_\infty(z) - \sum_{k=1}^{n} \psi_k(z),$$

which is meromorphic on $\mathbb{C}$, with potential singularities only at $z_1, ..., z_n$ and ∞.

We now show that each of these singularities is removable. First, fix $j \in \{1, ..., n\}$ arbitrarily. Since the poles are isolated, there exists $\varepsilon_j > 0$ such that the punctured disk $D^*(z_j, \varepsilon_j) = \{z : 0 < |z - z_j| < \varepsilon_j\}$ contains no other poles z_k for $k \neq j$.

1. Since $f(z) - \psi_j(z)$ is the holomorphic part of the Laurent expansion at z_j, it is holomorphic on $D(z_j, \varepsilon_j)$ (including at z_j).
2. $\sum_{k \neq j} \psi_k(z)$ is holomorphic on $D(z_j, \varepsilon_j)$, as each ψ_k has its singularity elsewhere.
3. $\psi_\infty(z)$ is a polynomial, hence entire.

Thus,

$$\Phi(z) = [f(z) - \psi_j(z)] - \psi_\infty(z) - \sum_{k \neq j} \psi_k(z)$$

is holomorphic on $D(z_j, \varepsilon_j)$, including at z_j. Therefore, we can define $\Phi(z_j)$ to make Φ holomorphic at z_j.

Since $f(z) - \psi_\infty(z)$ is the holomorphic part of the expansion at ∞, consisting of terms with nonpositive powers of z, $\lim_{z \to \infty} f(z) - \psi_\infty(z)$ exists and is finite. Additionally, each $\psi_k(z)$ consists of negative powers of $z - z_k$, so $\lim_{z \to \infty} \psi_k(z) = 0$ for each k, and thus $\lim_{z \to \infty} \sum_{k=1}^{n} \psi_k(z) = 0$. Therefore, $\lim_{z \to \infty} \Phi(z)$ exists and is finite, so ∞ is a removable singularity of Φ. Without the finite singularities at each z_k, Φ is entire. Since Φ has a finite limit at ∞, it is bounded on $\mathbb{C}$. By Liouville's theorem, $\Phi(z) \equiv c$ for some constant c.

Hence,

$$f(z) = c + \psi_\infty(z) + \sum_{k=1}^{n} \psi_k(z).$$

The right-hand side is a sum of a constant, a polynomial, and finitely many principal parts (each a rational function with a single pole), so f is rational.□

If $z = \infty$ is not a pole or removable singularity of a meromorphic function $f(z)$, then it is either an essential singularity or an accumulation point of poles. In this case, f is not rational and is known as a *transcendental meromorphic function.*

4.4 Further Properties of Meromorphic and Entire Functions

Theorem 4.4.1: Let $U \subseteq \mathbb{C}$ be a region and $f : U \to \mathbb{C}$ be meromorphic. Let $\gamma \subset U$ be a positively oriented Jordan curve that is null-homotopic in U. If f has no zeros on γ, then f has finitely many zeros and poles in the region bounded by γ. Denote the zeros of f in the bounded region by $a_1, ..., a_k$ with respective multiplicities $\alpha_1, ..., \alpha_k$, and the poles by $b_1, ..., b_m$ with respective orders $\beta_1, ..., \beta_m$. Let ψ be any function holomorphic on a neighborhood of the closure of the bounded region. Then

$$\frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{\psi(z) f'(z)}{f(z)} \,\mathrm{d}z = \sum_{i=1}^{k} \alpha_i \psi(a_i) - \sum_{j=1}^{m} \beta_j \psi(b_j).$$

Proof: Choose disks $D(a_i, \varepsilon_i)$ with pairwise disjoint closures around each zero a_i and $D(b_j, \varepsilon'_j)$ around each pole b_j, with $\varepsilon_i, \varepsilon'_j > 0$ sufficiently small so that these disks are contained in $\operatorname{int}(\gamma)$, disjoint from γ, and contained in the neighborhood where ψ is holomorphic. The function

$$g(z) = \frac{\psi(z) f'(z)}{f(z)}$$

is holomorphic on

$$\operatorname{int}(\gamma) \setminus \left(\bigcup_{i=1}^{k} D(a_i, \varepsilon_i) \cup \bigcup_{j=1}^{m} D(b_j, \varepsilon'_j) \right),$$

since ψ is holomorphic there, f is meromorphic with no other singularities, and $f \neq 0$ on γ. The oriented boundary of this punctured domain is $\gamma^+ \cup \bigcup_{i=1}^{k} \partial D(a_i, \varepsilon_i)^- \cup \bigcup_{j=1}^{m} \partial D\big(b_j, \varepsilon_j'\big)^-$. By Cauchy–Goursat (Theorem 3.1.7),

$$\oint_{\gamma^+} g(z)\,\mathrm{d}z + \sum_{i=1}^{k} \oint_{\partial D(a_i,\varepsilon_i)^-} g(z)\,\mathrm{d}z + \sum_{j=1}^{m} \oint_{\partial D\left(b_j,\varepsilon_j'\right)^-} g(z)\,\mathrm{d}z = 0.$$

Thus,

$$-\oint_{\gamma} g(z)\,\mathrm{d}z = \sum_{i=1}^{k} \oint_{\partial D(a_i,\varepsilon_i)^+} g(z)\,\mathrm{d}z + \sum_{j=1}^{m} \oint_{\partial D\left(b_j,\varepsilon_j'\right)^+} g(z)\,\mathrm{d}z.$$

Near each zero a_i, write $f(z) = (z - a_i)^{\alpha_i} h(z)$ where h is holomorphic at a_i with $h(a_i) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \frac{\alpha_i}{z - a_i} + \frac{h'(z)}{h(z)},$$

so

$$g(z) = \psi(z)\left(\frac{\alpha_i}{z - a_i} + \frac{h'(z)}{h(z)}\right).$$

Then,

$$\oint_{\partial D(a_i,\varepsilon_i)} g(z)\,\mathrm{d}z = \oint_{\partial D(a_i,\varepsilon_i)} \psi(z)\left(\frac{\alpha_i}{z - a_i} + \frac{h'(z)}{h(z)}\right)\mathrm{d}z = 2\pi\mathrm{i}\alpha_i\psi(a_i),$$

where the first term has been reduced by the Cauchy–Goursat Formula (Theorem 3.1.8) and the second integral vanishes by the Cauchy–Goursat Theorem (Theorem 3.1.7).

Near a pole b_j, write $f(z) = \big(z - b_j\big)^{-\beta_j} k(z)$ where k is holomorphic at b_j with $k\big(b_j\big) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = -\frac{\beta_j}{z - b_j} + \frac{k'(z)}{k(z)},$$

so

$$g(z) = \psi(z)\left(-\frac{\beta_j}{z - b_j} + \frac{k'(z)}{k(z)}\right).$$

A similar calculation yields that

$$\oint_{\partial D\left(b_j,\varepsilon_j'\right)} g(z)\,\mathrm{d}z = -2\pi \mathrm{i}\beta_j\psi\left(b_j\right).$$

Combining these,

$$\begin{aligned}\oint_\gamma \frac{\psi(z)f'(z)}{f(z)}\,\mathrm{d}z &= \sum_{i=1}^{k} 2\pi\mathrm{i}\alpha_i\psi(a_i) - \sum_{j=1}^{m} 2\pi\mathrm{i}\beta_j\psi\left(b_j\right) \\ &= 2\pi\mathrm{i}\left(\sum_{i=1}^{k} \alpha_i\psi(a_i) - \sum_{j=1}^{m}\beta_j\psi\left(b_j\right)\right).\end{aligned} \qquad \square$$

Theorem 4.4.2 *(Argument Principle)*: Let $U \subseteq \mathbb{C}$ be a region and $f : U \to \mathbb{C}$ be meromorphic. Let $\gamma \subset U$ be a simple, closed, positively oriented curve that is null-homotopic in U. If f has no zeros or poles on γ, then f has finitely many zeros and poles in the region bounded by γ, and the number of zeros, k, minus the number of poles, k', counting multiplicities and orders, is given by

$$k - k' = \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

Let Γ be the image of γ under the map $w = f(z)$. Then $k - k' = \mathrm{Ind}_\Gamma(0)$.

Proof: By Theorem 4.4.1 for $\psi \equiv 1$,

$$\frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = k - k'.$$

Parametrize Γ by $w = f(z)$. Then $\mathrm{d}w = f'(z)\,\mathrm{d}z$, and

$$k - k' = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}w}{w} = \mathrm{Ind}_\Gamma(0). \qquad \square$$

4.4.1 The Group of Holomorphic Automorphisms on $\mathbb{C}$

In complex analysis, three main sets of interest are $\mathbb{D}$, $\mathbb{C}$, and $\hat{\mathbb{C}}$. We will now find $\mathrm{Aut}(\mathbb{C})$.

Theorem 4.4.1.1 *(The Holomorphic Automorphism Group on $\mathbb{C}$)*: $\forall f \in \mathrm{Aut}(\mathbb{C})$, f is linear and non-constant. In other words, $\exists a \in \mathbb{C} \setminus \{0\}$ and $\exists b \in \mathbb{C}$ such that

$$f(z) = az + b.$$

Proof: First, assume that ∞ is not an essential singularity of $f(z)$, which we will prove later. Then ∞ must be a pole by trichotomy, as a removable singularity implies boundedness (Proposition 4.3.1). Therefore, $f(z)$ is a polynomial of degree m, where $m \in \mathbb{N}$.

Since $f^{-1} \in \operatorname{Aut}(\mathbb{C})$, it is true that $\left(f^{-1}\right)'$ is entire. Since

$$(f^{-1})' = \frac{1}{f'(f^{-1})},$$

it follows that f' has no zeros in $\mathbb{C}$. By the Fundamental Theorem of Algebra (Theorem 3.3.1), if $m > 1$, then f' has a complex zero, which is a contradiction. Hence, f must be linear, and all functions in $\operatorname{Aut}(\mathbb{C})$ are in the form of $az + b$, where $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$ are constants. In other words, any holomorphic automorphism on $\mathbb{C}$ is a composition of a rotation, a dilation, and a translation.

We will now prove the primary assumption; the singularity at $z = \infty$ cannot be an essential singularity of $f(z)$. Let $w \in \mathbb{C}$ be arbitrary. Then by the Casorati–Weierstrass Theorem (Theorem 4.2.3), $\forall \varepsilon > 0$ and $\forall R > 0$, $\exists |z| > R$ such that $|f(z) - w| < \varepsilon$. Equivalently, $\forall R > 0$, $\exists \zeta \in D(w, \varepsilon)$ such that $\left|f^{-1}(\zeta)\right| > R$. Since f^{-1} is continuous on $\overline{D(w, \varepsilon)}$ by holomorphy, by Theorem 1.2.13, it is bounded, which is a contradiction. □

4.4.2 The Group of Meromorphic Automorphisms on $\hat{\mathbb{C}}$

It is generally common to consider a meromorphic function as a function in the form of $f : U \to \hat{\mathbb{C}}$. Let $\operatorname{Aut}\left(\hat{\mathbb{C}}\right)$ denote the group of meromorphic automorphisms on $\hat{\mathbb{C}}$.

To make more profound conclusions on the structure of $\operatorname{Aut}\left(\hat{\mathbb{C}}\right)$, we will introduce certain concepts from group theory.

Definition 4.4.2.1 *(Coset)*: Let G be a group, and let $H \leq G$ be a subgroup (operation denoted by juxtaposition). Then the *left coset* of H in G with respect to $g \in G$ is defined as

$$gH = \{gh : h \in H\}.$$

The *right coset* is defined as

$$Hg = \{hg : h \in H\}.$$

The subgroup H is *normal* iff the left and right cosets are equal. The notation $H \trianglelefteq G$ is used to represent a normal subgroup. Cosets, like groups and sets, are unordered.

Theorem 4.4.2.1: Let G be a group and $N \leq G$ a subgroup. The set of left cosets

$$G/N = \{gN : g \in G\}$$

admits a group structure with operation

$$(gN)(hN) = (gh)N$$

if and only if N is a normal subgroup of G (i.e. $N \trianglelefteq G$).

Proof: We prove the two implications separately.

1. *If $N \trianglelefteq G$ then G/N is a group.*

   Assume N is normal, $N \trianglelefteq G$, so $gNg^{-1} = N$ for every $g \in G$ (equivalently $g^{-1}Ng = N$). Define a product on G/N by

   $$(gN)(hN) = (gh)N.$$

   We now verify that this product is well-defined: if $gN = g'N$ and $hN = h'N$ then we need $(gh)N = (g'h')N$. Since $gN = g'N$, there exists $n_1 \in N$ with $g' = gn_1$, and since $hN = h'N$ there exists $n_2 \in N$ with $h' = hn_2$. Then

   $$g'h' = (gn_1)(hn_2) = g(n_1h)n_2 = gh(h^{-1}n_1h)n_2.$$

   Because N is normal we have $h^{-1}n_1h \in N$, and $n_2 \in N$, so $(h^{-1}n_1h)n_2 \in N$. Hence $g'h' \in (gh)N$, meaning that $(g'h')N = (gh)N$. Thus the product is well-defined.

   Associativity follows from associativity in G:

   $$((gN)(hN))(kN) = (ghk)N = (gN)((hN)(kN)).$$

   The identity is $eN = N$, since $(eN)(gN) = (eg)N = gN$ and similarly on the other side. The inverse of gN is $g^{-1}N$, because $(gN)(g^{-1}N) = (gg^{-1})N = N$. Thus G/N is a group.

2. *If G/N can be given a group structure via the coset multiplication, then $N \trianglelefteq G$.*

   Fix $g \in G$ and $n \in N$ arbitrarily. By assumption, we have

   $$(gN)(nN)(g^{-1}N) = (gng^{-1})N.$$

   Because $nN = eN$, we also have

   $$(gN)(nN)(g^{-1}N) = (gN)(eN)(g^{-1}N) = (gg^{-1})N = N,$$

implying that $\left(gng^{-1}\right)N = N$, and hence $gng^{-1} \in N$ for any $g \in G$, $n \in N$. Hence, $gNg^{-1} \subseteq N$. Now replacing g with g^{-1} and rearranging yields $n \in gNg^{-1}$, or that $N \subseteq gNg^{-1}$. Therefore, N is normal.

□

Under the normality of N, the group G/N is known as the *quotient group* of G by N.

Remark: Every subgroup of an abelian group is normal.

Definition 4.4.2.2 *(Group Homomorphism)*: Let $(G, \cdot)$ and $(H, *)$ be groups. A function $\varphi : G \to H$ is said to be a *group homomorphism* if

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) * \varphi(g_2) \quad \forall g_1, g_2 \in G.$$

Definition 4.4.2.3 *(Group Isomorphism)*: A group homomorphism $\varphi : G \to H$ is called an *isomorphism* if it is bijective.

If there exists an isomorphism between two groups G and H, they are said to be *isomorphic*, denoted by $G \cong H$. The utility of groups allows us to classify them according to their structure: if two groups are isomorphic, they are essentially the same from a group-theoretic perspective. This viewpoint lets us replace complicated groups with simpler, isomorphic ones, and study their properties without loss of generality.

Let us now examine $\operatorname{Aut}\left(\hat{\mathbb{C}}\right)$. Let $f(z) \in \operatorname{Aut}\left(\hat{\mathbb{C}}\right)$ such that $f(\infty) = \infty$. It follows that f maps $\mathbb{C}$ to $\mathbb{C}$ bijectively and $f \in \operatorname{Aut}(\mathbb{C}) < \operatorname{Aut}\left(\hat{\mathbb{C}}\right)$. Therefore, $f(z)$ has the form $az + b$, where $a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$ are constants.

Let $f(z) \in \operatorname{Aut}\left(\hat{\mathbb{C}}\right)$ such that $f(\infty) \neq \infty$. Then,

$$g(z) = \frac{1}{f(z) - f(\infty)}$$

is in $\operatorname{Aut}\left(\hat{\mathbb{C}}\right)$ and $g(\infty) = \infty$. By the property above, $g(z) = cz + d$ for some complex d and nonzero c. Hence,

$$f(z) = \frac{f(\infty)(cz + d) + 1}{cz + d}.$$

Let $a = cf(\infty)$ and $b = df(\infty) + 1$. Then

$$f(z) = \frac{az + b}{cz + d}.$$

In this specific construction, $ad - bc = -c \neq 0$. Let the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ correspond to this transformation, where for any nonzero scalar k, $k\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ corresponds to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Therefore, we can arbitrarily pick $ad - bc$ to be 1.

Therefore, there exists a one-to-one correspondence between $\operatorname{Aut}\left(\hat{\mathbb{C}}\right)$ and the group under matrix multiplication of

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = 1\right\} / \{\pm \mathbf{I}\}.$$

The quotient group is taken because the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ corresponds to the same transformation as $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$. This group, denoted by

$$\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C}) / \{\pm \mathbf{I}\} \cong \operatorname{Aut}\left(\hat{\mathbb{C}}\right),$$

is known as the *projective special linear group* of order 2, and is isomorphic to the *group of Möbius transformations*, consisting of all complex linear fractional transformations.

Therefore, any meromorphic automorphism on $\hat{\mathbb{C}}$ is a composition of rotations, dilations, translations, and inversions. We will now state this formally:

Theorem 4.4.2.2 *(The Meromorphic Automorphism Group on $\hat{\mathbb{C}}$)*: $\forall f \in \operatorname{Aut}\left(\hat{\mathbb{C}}\right)$, f is a Möbius transformation. In other words, $\exists a, b, c, d \in \mathbb{C}$ satisfying $ad - bc \neq 0$ such that

$$f(z) = \frac{az + b}{cz + d}.$$

Moreover, every such Möbius transformation is in $\operatorname{Aut}\left(\hat{\mathbb{C}}\right)$.

The group of holomorphic automorphisms on $\mathbb{D}$, or $\operatorname{Aut}(\mathbb{D})$, is also a subgroup of $\operatorname{Aut}\left(\hat{\mathbb{C}}\right)$.

Proposition 4.4.2.1: Suppose we have two Möbius transformations represented by the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then their composition is a Möbius transformation represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

Proof: From simple substitution, we have

$$\frac{a\frac{ez+f}{gz+h} + b}{c\frac{ez+f}{gz+h} + d} = \frac{aez + af + bgz + bh}{cez + cf + dgz + dh} = \frac{(ae + bg)z + (af + bh)}{(ce + dg)z + (cf + dh)},$$

which corresponds to the product $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$. □

We have now introduced three of the most important regions in complex analysis: $\mathbb{D}$, $\mathbb{C}$, and $\hat{\mathbb{C}}$. Their importance will be later explained by the Uniformization Theorem (@ thm:uniformization).

4.4.3 The Construction of Entire and Meromorphic Functions

It is common knowledge in algebra that any polynomial can be factored into linear factors. When can this factorization be extended to transcendental entire functions?

We will start by introducing the concept of *infinite products.* Let

$$\prod_{k=1}^{n}(1+u_k)$$

be an infinite product. If the limit

$$\lim_{n\to\infty}\prod_{k=1}^{n}(1+u_k)$$

exists and is finite, then the infinite product is said to be *convergent.*

For $x \in \mathbb{R}_{\geq 0}$, since $\mathrm{e}^x \geq x$ and $\mathrm{e}^0 = 1$, we can integrate over $[0, x]$ to get that $\mathrm{e}^x \geq x + 1$. Therefore,

$$\begin{aligned}\exp\left(\sum_{k=1}^{n}|u_k|\right) \geq \prod_{k=1}^{n}(1+|u_k|) &= 1+\sum_{k=1}^{n}|u_k| \\ &+ \sum_{\substack{j,k\in\{1,\dots,n\}\\ j<k}}|u_j u_k| + \cdots + \prod_{k=1}^{n}|u_k| > \sum_{k=1}^{n}|u_k|.\end{aligned}$$

Since the convergence of $\sum_{k=1}^{\infty}|u_k|$ is the same as that of $\exp\left(\sum_{k=1}^{\infty}|u_k|\right)$, it follows that the convergence of $\sum_{k=1}^{\infty}|u_k|$ is equivalent to that of $\prod_{k=1}^{\infty}(1+|u_k|)$. If $\sum_{k=1}^{\infty}|u_k|$ is convergent, then $\prod_{k=1}^{\infty}(1+u_k)$ is *absolutely convergent.* As with the order of summing an absolutely convergent series is unimportant, we may also rearrange terms in an absolutely convergent infinite product.

Similar to series, absolute convergence is a stronger condition than convergence:

Lemma 4.4.3.1: An absolutely convergent infinite product is convergent.

Proof: Let $\{u_k\}_{k\in\mathbb{N}}$ be a complex sequence such that $\sum_{k=1}^{\infty}|u_k|$ is convergent. Then $\prod_{k=1}^{\infty}(1+u_k)$ is absolutely convergent. Let Q_n denote the partial products of $\prod_{k=1}^{n}(1+|u_k|)$ and let P_n denote the partial products of $\prod_{k=1}^{n}(1+u_k)$. By the Cauchy Criterion (Theorem 1.2.12), we have that $\forall\varepsilon > 0$, $\exists N \in$

$\mathbb{N}$ such that $\forall n > m > N$, $|Q_m - Q_n| < \varepsilon$. Let us now analyze the absolute difference between P_n and P_m:

$$\begin{aligned} |P_n - P_m| &= \left| \prod_{k=1}^{n} (1 + u_k) - \prod_{k=1}^{m} (1 + u_k) \right| \\ &= \left| \prod_{k=1}^{m} (1 + u_k) \prod_{k=m+1}^{n} (1 + u_k) - \prod_{k=1}^{m} (1 + u_k) \right| \\ &= \prod_{k=1}^{m} |1 + u_k| \cdot \left| \prod_{k=m+1}^{n} (1 + u_k) - 1 \right| \\ &\leq \prod_{k=1}^{m} (1 + |u_k|) \cdot \left| \prod_{k=m+1}^{n} (1 + |u_k|) - 1 \right| \\ &= |Q_n - Q_m| < \varepsilon, \end{aligned}$$

which therefore satisfies Theorem 1.2.12. □

We will now provide the following assertions on the *locally uniform convergence* of infinite products:

Lemma 4.4.3.2: Let $U \subseteq \mathbb{C}$ be open and connected. Suppose $\sum_{k=1}^{\infty} f_k(z)$ uniformly converges on compact subsets of U such that each f_k is holomorphic on U. Then the infinite product

$$\prod_{k=1}^{\infty} \exp(f_k(z))$$

is uniformly convergent on compact subsets of U.

Proof: Let K be an arbitrary compact subset of U. Since $\sum_{k=1}^{\infty} f_k(z)$ converges uniformly on K, it follows that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > m > N$, $\left| \sum_{k=m+1}^{n} f_k(z) \right| < \varepsilon$ for all $z \in K$. Additionally, we have

$$\left| \prod_{k=1}^{n} \exp(f_k(z)) - \prod_{k=1}^{m} \exp(f_k(z)) \right| = \left| \exp\left(\sum_{k=1}^{n} f_k(z) \right) - \exp\left(\sum_{k=1}^{m} f_k(z) \right) \right|.$$

By Theorem 4.1.1, the uniform limit $\sum_{k=1}^{\infty} f_k(z)$ is holomorphic on U. By continuity and Theorem 1.2.13, this limit is bounded on K. It follows that each partial sum is uniformly bounded on K. Since the exponential function is Lipschitz continuous on compact subsets of $\mathbb{C}$, there exists a finite constant $M > 0$ such that

$$\left|\exp\left(\sum_{k=1}^{n} f_k(z)\right) - \exp\left(\sum_{k=1}^{m} f_k(z)\right)\right| \leq M\left|\sum_{k=m+1}^{n} f_k(z)\right| < M\varepsilon. \quad \square$$

Remark: Uniform convergence on compact subsets is also known as *compact convergence*. In the case of $\mathbb{C}$ (or in any topological space such that every point has a compact neighborhood), compact convergence is equivalent to *locally uniform convergence*.

We also have:

Lemma 4.4.3.3: Let $U \subseteq \mathbb{C}$ be open and connected. Suppose $\sum_{k=1}^{\infty} |f_k(z)|$ is uniformly convergent on compact subsets of U such that each f_k is holomorphic on U. Then the infinite product

$$\prod_{k=1}^{\infty} (1 + f_k(z))$$

is uniformly convergent on compact subsets of U to a holomorphic function, which vanishes only at a point z if and only if $f_k(z) = -1$ for some $k \in \mathbb{N}$. The multiplicity at each such zero z is the sum of the multiplicities of $1 + f_k$ at z for all k satisfying $f_k(z) = -1$.

Proof: Let $K \subset U$ be an arbitrary compact set. By the uniform convergence of $\sum_{k=1}^{\infty} |f_k(z)|$ on K, it follows that the uniform limit is continuous by the Uniform Limit Theorem (Theorem 2.3.5). By continuity on a compact set, it follows that the limit is bounded by some constant M'. Additionally, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $\sum_{k=1}^{n} |f_k(z)| < M' + \varepsilon$. It follows that the partial sums are uniformly bounded on K by

$$M = \max\left\{\max_{1 \leq k \leq N} \max_{z \in K} |f_k(z)|, M' + \varepsilon\right\}.$$

Similarly, by earlier discussion of infinite products, we have

$$F_n(z) = \prod_{k=1}^{n} (1 + |f_k(z)|) \leq \exp\left(\sum_{k=1}^{n} |f_k(z)|\right) \leq \mathrm{e}^{M},$$

or in other words, the partial products are uniformly bounded on K. Let $0 < \varepsilon < 1$ be arbitrary. By definition, there exists $N \in \mathbb{N}$ such that $\forall n > m > N$, $\left|\sum_{k=m+1}^{n} f_k(z)\right| < \varepsilon$ for all $z \in K$. The difference between the non-absolute partial products satisfies

$$\begin{aligned}\left|\prod_{k=1}^{n}(1+f_k(z))-\prod_{k=1}^{m}(1+f_k(z))\right| &\leq \left|\prod_{k=1}^{m}(1+f_k(z))\right|\left|\prod_{k=m+1}^{n}(1+f_k(z))-1\right| \\ &\leq |F_m(z)|\left|\exp\left(\sum_{k=m+1}^{n}|f_k(z)|\right)-1\right| \\ &\leq \mathrm{e}^M(\mathrm{e}^\varepsilon-1),\end{aligned}$$

where the second inequality can be easily verified by expanding the product $\prod_{k=m+1}^{n}(1+f_k(z))-1$ and the triangle inequality.

Since $\mathrm{e}^\varepsilon-1\to 0^+$, it follows that

$$F(z)=\prod_{k=1}^{\infty}(1+f_k(z))$$

is uniformly convergent on K. Let $\xi\in U$ be a point satisfying $F(\xi)=0$. Since there exists an $m\in\mathbb{N}$ such that

$$\prod_{k=m+1}^{\infty}(1+f_k(z))$$

is non-vanishing at $z=\xi$, and from the fact that

$$F(z)=\prod_{k=1}^{m}(1+f_k(z))\cdot\prod_{k=m+1}^{\infty}(1+f_k(z)),$$

we can analyze the zeros of the finite product to obtain the conclusion. □

We will now study the construction of an entire function $f(z)$ via its zeros. We have the following cases:

1 If f has no zeros in $\mathbb{C}$, then the function defined by $z\mapsto\frac{f'(z)}{f(z)}$ is entire, so it is the derivative of an entire function $\varphi(z)$. Therefore, the function defined by $z\mapsto f(z)\mathrm{e}^{-\varphi(z)}$ has the vanishing derivative

$$\frac{\mathrm{d}f(z)\mathrm{e}^{-\varphi(z)}}{\mathrm{d}z}=f'(z)\mathrm{e}^{-\varphi(z)}-\varphi'(z)f(z)\mathrm{e}^{-\varphi(z)}=0.$$

It follows that $f(z)\mathrm{e}^{-\varphi(z)}$ is constant, and therefore $f(z)=c\exp(\varphi(z))$ for some constant $c\in\mathbb{C}$. Since $\varphi(z)$ is entire, it follows that $f(z)$ is also entire. Absorb the constant c into $\varphi(z)$, and $f(z)=\exp(\varphi(z))$.

2 If f is entire and has finitely many zeros in $\mathbb{C}$, namely $a_0=0,a_1,a_2,\ldots,a_n$ with respective multiplicities $m_0,m_1,\ldots,m_n$ (if 0 is not a zero, treat $m_0=0$), then at each zero a_k, it has the local Taylor expansion

$$f(z) = \sum_{j=m_k}^{\infty} c_j (z - a_k)^j,$$

where $c_{m_k} \neq 0$. Therefore, we can divide $f(z)$ by $(z - a_k)^j$ to obtain a new entire function with no additional zeros and no zero at a_k. Repeating this for every zero, we can define

$$\psi(z) = \frac{f(z)}{p(z)},$$

which is entire and has no zeros, where

$$p(z) = z^{m_0} \left(1 - \frac{z}{a_1}\right)^{m_1} \dots \left(1 - \frac{z}{a_n}\right)^{m_n}.$$

We write $p(z)$ in the above form rather than that of $z^{m_0} \prod_{k=1}^{n} (z - a_k)^{m_k}$ as we aim to generalize the construction to infinite products to study convergence. By the non-vanishing case above, $\psi(z) = \exp(\varphi(z))$ for some entire function $\varphi(z)$. Therefore, we can write

$$f(z) = p(z) \exp(\varphi(z)), \tag{4.4.1}$$

where $p(z)$ is a polynomial with zeros at a_k with respective multiplicities m_k. The entire functions $p(z)$ and $f(z)$ both have the same zeros with matching multiplicities.

3 If $f(z)$ is entire and has infinitely many zeros such that f is not identically zero, then it follows that f has countably many zeros (since the zeros are isolated). Let the zeros be indexed by $\mathbb{N}$, namely $a_1, a_2, \dots$. Without loss of generality, assume that $\forall n \in \mathbb{N}$, $0 < |a_n| \leq |a_{n+1}|$ (repeated elements representing multiplicities), and $\lim_{n\to\infty} a_n = \infty$. The case for a zero at 0 will be treated differently.

There exists a positive integer sequence $p_1, p_2, \dots$ such that for every positive and finite R, $\sum_{n=1}^{\infty} \left|\frac{R}{a_n}\right|^{p_n+1}$ converges. For example, let $p_n = n$, and for sufficiently large n, $\frac{R}{|a_n|} < 1$ and the series is convergent. Consider the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \dots + \frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}\right). \tag{4.4.2}$$

Let

$$\begin{aligned} P_p(z) &= z + \frac{1}{2}z^2 + \cdots + \frac{1}{p}z^p \\ Q_p(z) &= \log(1-z) + P_p(z) \\ E_p(z) &= \exp\bigl(Q_p(z)\bigr) = (1-z)\exp\bigl(P_p(z)\bigr). \end{aligned} \tag{4.4.3}$$

Therefore, we can rewrite (4.4.2) as

$$\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

The expression in (4.4.3) is known as the p-th *Weierstrass elementary factor*.

By $\varepsilon - -N$, for a fixed $R > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n| > 2R$. Consider the product $\prod_{n=N}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$. For $z \in \overline{D(0,R)}$ and $n \geq N$, we have $\left|\frac{z}{a_n}\right| \leq \frac{1}{2}$. The Taylor expansion

$$\log(1-w) = -\sum_{k=1}^{\infty} \frac{w^k}{k}$$

has a convergence disk of $D(0,1)$. Then,

$$\begin{aligned} \left|Q_{p_n}\left(\frac{z}{a_n}\right)\right| &= \left|-\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{z}{a_n}\right)^k + \sum_{j=1}^{p_n} \frac{1}{j}\left(\frac{z}{a_n}\right)^j\right| \\ &\leq \sum_{k=p_n+1}^{\infty} \frac{1}{k}\left|\frac{z}{a_n}\right|^k \\ &\leq \sum_{k=p_n+1}^{\infty} \left|\frac{z}{a_n}\right|^k = \left|\frac{z}{a_n}\right|^{p_n+1} \frac{1}{1-\left|\frac{z}{a_n}\right|} \\ &\leq 2\left|\frac{R}{a_n}\right|^{p_n+1}. \end{aligned} \tag{4.4.4}$$

By the definition of $\{p_n\}_{n\in\mathbb{N}}$, the series $2\sum_{n=1}^{\infty}\left|\frac{R}{a_n}\right|^{p_n+1}$ converges. Therefore, $\sum_{n=1}^{\infty} Q_{p_n}\left(\frac{z}{a_n}\right)$ is uniformly and absolutely convergent on $\overline{D(0,R)}$ by the Weierstrass M-Test (Theorem 2.3.2). We then get that

$$\prod_{n=N}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right) = \exp\left(\sum_{n=N}^{\infty} Q_{p_n}\left(\frac{z}{a_n}\right)\right),$$

and it uniformly converges on $\overline{D(0,R)}$ to a nonzero holomorphic function $f(z)$ on $D(0,R)$ by Lemma 4.4.3.2, Theorem 4.1.1, and Theorem 3.3.5.

The zeros of

$$\prod_{n=1}^{N-1} E_{p_n}\left(\frac{z}{a_n}\right)$$

are $a_1, ..., a_{N-1}$ and lie in $\overline{D(0, 2R)}$. To prove the absolute convergence of $\prod_{n=N}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ on $\overline{D(0, R)}$, we will show that $\sum_{n=N}^{\infty}\left|E_{p_n}\left(\frac{z}{a_n}\right) - 1\right|$ is convergent. Trivially, when $\zeta \in \overline{\mathbb{D}}$, we have

$$|\exp(\zeta) - 1| \leq \exp(|\zeta|) - 1 \leq (\mathrm{e} - 1)|\zeta|.$$

By (4.4.4) above, we get that $\left|Q_{p_n}\left(\frac{z}{a_n}\right)\right| \leq 1$ when $n \geq N$.

Therefore, we have

$$\begin{aligned}\left|E_{p_n}\left(\frac{z}{a_n}\right) - 1\right| &= \left|\exp\left(Q_{p_n}\left(\frac{z}{a_n}\right)\right) - 1\right| \\ &\leq (\mathrm{e} - 1)\left|Q_{p_n}\left(\frac{z}{a_n}\right)\right| \leq 2(\mathrm{e} - 1)\left|\frac{R}{a_n}\right|^{p_n+1},\end{aligned}$$

which has a convergent series by definition. Therefore, $\prod_{n=N}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ is absolutely convergent on $\overline{D(0, R)}$. Letting $R \to \infty$, we obtain the following result:

Theorem 4.4.3.1 *(Weierstrass Product Theorem)*: Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of nonzero complex numbers satisfying $a_n \to \infty$ as $n \to \infty$ and $0 < |a_n| \leq |a_{n+1}|$ (equality of a_n and a_{n+1} treated as multiplicities) for all n. Then there exists a sequence $\{p_n\}_{n \in \mathbb{N}}$ of nonnegative integers such that $\forall R > 0$, $\sum_{n=1}^{\infty}\left|\frac{R}{a_n}\right|^{p_n+1}$ converges. For such a prescribed sequence, the function

$$f(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right) \tag{4.4.5}$$

defines an entire function with zeros at each element of the sequence of multiplicities matching the number of times an element is repeated. Moreover, the product converges uniformly on any compact disk $\overline{D(0, R)}$.

The following result is then apparent:

Theorem 4.4.3.2 *(Weierstrass Factorization Theorem)*: Suppose $f(z)$ is an entire function. Let $\{a_n\}_{n \in \mathbb{N}}$ be the sequence of all nonzero zeros of f satisfying $a_n \to \infty$ as $n \to \infty$ and $0 < |a_n| \leq |a_{n+1}|$ (equality of a_n and a_{n+1} treated as multiplicities) for all n. Let m be the multiplicity of $f(z)$ at $z = 0$ (let $m = 0$ if there is no zero at 0). Then there exists a sequence

$\{p_n\}_{n\in\mathbb{N}}$ of nonnegative integers such that $\forall R > 0$, $\sum_{n=1}^{\infty} \left|\frac{R}{a_n}\right|^{p_n+1}$ converges. Then, we can write

$$f(z) = z^m \mathrm{e}^{\varphi(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right) \tag{4.4.6}$$

on $D(0, R)$, where $E_p(z)$ is the p-th Weierstrass elementary factor defined in (4.4.3) and $\varphi(z)$ is an entire function. The infinite product converges uniformly on $\overline{D(0, R)}$ and converges absolutely on $\mathbb{C}$. If we let $p_n = n$, we can write

$$f(z) = z^m \mathrm{e}^{\varphi(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{z}{a_n}\right)^n\right).$$

Proof: By the Weierstrass Product Theorem, construct $\psi(z)$ to be entire and have zeros at $\{a_n\}_{n\in\mathbb{N}}$. Thus, $z^m\psi(z)$ and $f(z)$ have the same zeros and corresponding multiplicities. Then the function

$$z^m \frac{\psi(z)}{f(z)}$$

has removable singularities on all of $\{a_n\}_{n\in\mathbb{N}} \cup \{0\}$ and has an analytic continuation (Theorem 3.2.6) to an entire and non-vanishing function. Therefore, it can be written as

$$z^m \frac{\psi(z)}{f(z)} = \mathrm{e}^{\phi(z)},$$

where ϕ is entire. Let $\varphi = -\phi$, and from rearrangement, we obtain (4.4.6).□

Corollary 4.4.3.2.1: Let f be meromorphic on $\mathbb{C}$. Then f can be written as the quotient of two entire functions.

Proof: Let $\varphi(z)$ be any entire function with zeros only at each pole of f (with multiplicities matching the order of each pole). If there are infinitely many poles, we can explicitly construct such a φ by the Weierstrass Product Theorem (Theorem 4.4.3.1). If there are finitely many poles, construct φ using (4.4.1). It follows that φf can be analytically continued on its removable singularities to an entire function $\phi(z)$ with the same zeros as $f(z)$. Hence,

$$f(z)\varphi(z) = \phi(z) \Longleftrightarrow f(z) = \frac{\phi(z)}{\varphi(z)},$$

which is an explicit construction. □

Therefore, any meromorphic function on $\mathbb{C}$ can be expressed as the quotient of two infinite products. Hence, any meromorphic function on $\mathbb{C}$ can be explicitly written in terms of its zeros and poles.

We will now study the construction of meromorphic functions from their poles and the principal parts of their Laurent expansions at each pole.

Suppose $n \in \mathbb{N}$ and $\{a_k\}_{k=1}^n \subset \mathbb{C}$ is a sequence of distinct values. Let $\{\psi_k(z)\}_{k=1}^n$ be a collection of functions in the form

$$\psi_k(z) = \sum_{j=m_k}^{p_k} \frac{c_{k,j}}{(z-a_k)^j}, \tag{4.4.7}$$

where $p_k \geq m_k$ are finite integer constants and $\left\{c_{k,j}\right\}$ are complex constants.

Suppose that $f(z)$ is meromorphic on $\mathbb{C}$ such that f has finitely many poles. Therefore, f has an isolated singularity at ∞. We have two cases:

1 If $z = \infty$ is a removable singularity or a pole, by the given proof of Theorem 4.3.1, we may construct $f(z)$ to have poles at each of $\{a_k\}_{k=1}^n$ such that the principal parts of f at each of $\{a_k\}_{k=1}^n$ are $\{\psi_k(z)\}_{k=1}^n$. It can be explicitly written as

$$f(z) = p(z) + \sum_{k=1}^{n} \psi_k(z)$$

(we can absorb the constant c into the polynomial ψ_∞ as used in the proof).

2 In the case that $f(z)$ is a transcendental meromorphic function with an isolated essential singularity at $z = \infty$, notice that the function defined by

$$\varphi(z) = f(z) - \sum_{k=1}^{n} \psi_k(z)$$

has removable singularities at each of $\{a_k\}_{k=1}^n$. Indeed, since the singularities are isolated, for a fixed k, $\exists \varepsilon_k > 0$ such that for any $j \neq k$, $a_j \notin D(a_k, \varepsilon_k)$. It follows that ψ_j is holomorphic on $D(a_k, \varepsilon_k)$. Notice that $f(z) - \psi_k$ is the holomorphic part of the Laurent expansion at a_k and is also holomorphic on the disk. Suppose f has ψ_k as the principal part of its Laurent expansion at a_k. Then $\varphi(z)$ is holomorphic on $D(a_k, \varepsilon_k)$. Since k was arbitrarily chosen, φ is entire and transcendental.

Therefore, f can be constructed by

$$f(z) = \varphi(z) + \sum_{k=1}^{\infty} \psi_k(z),$$

for a transcendental entire function $\varphi(z)$.

3 The existence of a transcendental meromorphic function f whose poles have an accumulation point at $z = \infty$ is the concern of the following theorem:

Theorem 4.4.3.3 *(Mittag–Leffler)*: Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence of distinct complex numbers such that $\forall n \in \mathbb{N}$, $|a_n| \leq |a_{n+1}|$ and $\lim_{n\to\infty} a_n = \infty$. Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a function sequence, each in the form of (4.4.7). Then the following hold.

First, a meromorphic function $f(z)$ on $\mathbb{C}$ can be constructed such that $\forall n \in \mathbb{N}$, f has a pole at a_n with principal part ψ_n at a_n.

Second, the function $f(z)$ satisfying the criteria above can be explicitly represented as

$$f(z) = \varphi(z) + \sum_{n=1}^{\infty}(\psi_n(z) - p_n(z)) \tag{4.4.8}$$

for some sequence of polynomials $\{p_n(z)\}$ and an arbitrary entire function $\varphi(z)$.

Proof: The classical proof for this theorem allows for a more explicit construction, as in (4.4.8). As for the existence statement, we can prove the first assertion by use of the $\overline{\partial}$-problem.

Fix $n \in \mathbb{N}$, and let U_n be an open neighborhood of a_n such that $\forall i, j \in \mathbb{N}$ where $i \neq j$, $\overline{U_i} \cap \overline{U_j} = \varnothing$. Let V_n be a neighborhood of a_n that is relatively compact in U_n. By Theorem 3.2.1.6, for each n, there is a C^∞ function φ_n satisfying

$$\varphi_n(z) = \begin{cases} 1 \text{ if } z \in \overline{V_n}, \\ 0 \text{ if } z \in \mathbb{C} \setminus U_n. \end{cases}$$

Let

$$u(z) = \sum_{k=1}^{\infty} \varphi_k(z)\psi_k(z),$$

which is an element of $C^\infty\left(\mathbb{C} \setminus \{a_k\}_{k\in\mathbb{N}}\right)$. For a fixed $n \in \mathbb{N}$, it is true that $u \equiv \psi_n$ on $\overline{V_n} \setminus \{a_n\}$. Hence, although u is not meromorphic, it does have the required principal part near each a_k. Let

$$\phi(z) = \begin{cases} \frac{\partial u}{\partial \bar{z}} & \text{if } z \in \mathbb{C} \setminus \{a_k\}_{k\in\mathbb{N}}, \\ 0 & \text{if } z \in \{a_k\}_{k\in\mathbb{N}}. \end{cases}$$

Since $\partial u/\partial \overline{z} \equiv \partial \psi_n/\partial \overline{z} \equiv 0$ and is C^∞ on $\overline{V_n} \setminus \{a_n\}$ and ϕ vanishes on $\{a_k\}_{k\in\mathbb{N}}$, $\phi \in C^\infty(\mathbb{C})$. By the discussion preceding Theorem 3.1.6, there exists a C^∞ function $v(z)$ such that $\frac{\partial v}{\partial \overline{z}} = \phi(z)$ on $\mathbb{C}$. Since ϕ is C^∞, it follows that v is also C^∞. Define $f(z) = u(z) - v(z)$. Then

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial u}{\partial \overline{z}} - \frac{\partial v}{\partial \overline{z}} = \phi(z) - \phi(z) = 0,$$

which implies that f is holomorphic on $\mathbb{C} \setminus \{a_k\}_{k\in\mathbb{N}}$. Since u has the desired principal part ψ_n at each a_n and v is C^∞ (and hence removable at each singularity), it follows that f is meromorphic on $\mathbb{C}$ with principal parts ψ_n at each a_n, as desired.

Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a positive sequence such that $\sum_{n=1}^{\infty} \varepsilon_n$ is convergent. Without loss of generality, let $a_1 = 0$ (if a_1 is not a pole, set $\psi_1 = 0$). Choose $p_1(z) = 0$ (this can actually be any arbitrary polynomial). Fix $n \geq 2$. Since ψ_n is a polynomial in terms of $\frac{1}{z-a_n}$ and has its only pole at $z = a_n$, $\psi_n(z)$ is holomorphic on $D(0, |a_n|)$ and can be written as

$$\psi_n(z) = \sum_{k=0}^{\infty} \frac{\psi_n^{(k)}(0)}{k!} z^k.$$

By Theorem 2.3.3, this series is uniformly convergent on $D\big(0, \big|\frac{a_n}{2}\big|\big)$. Hence, $\exists \lambda_n \in \mathbb{N}$ such that

$$\left| \psi_n(z) - \sum_{k=0}^{\lambda_n} \frac{\psi_n^{(k)}(0)}{k!} z^k \right| < \varepsilon_n.$$

Let

$$p_n(z) = \sum_{k=0}^{\lambda_n} \frac{\psi_n^{(k)}(0)}{k!} z^k.$$

Fix $R > 0$ and let $N \in \mathbb{N}$ depend on R such that $|a_n| > 2R$ for all $n > N$ and $|a_n| \leq 2R$ for all $n \leq N$. Therefore, $\forall n > N$, $R < \big|\frac{a_n}{2}\big|$. Then $\forall z \in D(0, R)$, we have

$$|\psi_n(z) - p_n(z)| < \varepsilon_n.$$

By the convergence of $\sum_{n=N+1}^{\infty} \varepsilon_n$, by the Weierstrass M-Test (Theorem 2.3.2), the series

$$\Phi_N(z) = \sum_{n=N+1}^{\infty} (\psi_n(z) - p_n(z)) \tag{4.4.9}$$

converges uniformly on $D(0,R)$. Since $z < R < \left|\frac{a_n}{2}\right| < |a_n|$ when $n > N$, the pole of $\psi_n(z)$, namely $z = a_n$, is not in $D(0,R)$ when $n > N$. By Theorem 4.1.1, (4.4.9) is holomorphic on $D(0,R)$. Let

$$\Psi(z) = \sum_{n=1}^{N}(\psi_n(z) - p_n(z)) + \Phi_N(z).$$

The poles of $\Psi(z)$ in $D(0,R)$ are all of the a_n with corresponding principal parts $\psi_n(z)$, where $n \in \mathbb{N}$ and $a_n \in D(0,R)$. Since R was arbitrarily chosen, Ψ has poles at each a_n with the corresponding principal part $\psi_n(z)$ on $\mathbb{C}$. Let $\varphi(z) = f(z) - \Psi(z)$ be analytically continued onto each of $\{a_n\}_{n\in\mathbb{N}}$. Then $\varphi(z)$ is an entire function (since the Laurent expansions of φ at each of $\{a_n\}_{n\in\mathbb{N}}$ vanish). By rearrangement, we obtain our desired result. □

The Mittag–Leffler Theorem (Theorem 4.4.3.3) can also be generalized as follows:

Theorem 4.4.3.4: Let $U \subset \mathbb{C}$ be an open set with a simple closed boundary ∂U and let $E = \{a_n\}_{n\in\mathbb{N}} \subset U$ be a sequence of distinct complex numbers whose accumulation points lie on ∂U. Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a sequence of functions in the form of (4.4.7). Then there exists a meromorphic function $f : U \to \mathbb{C}$ with poles at each a_n with principal parts ψ_n at each a_n.

Indeed, since $\partial U \cap U = \varnothing$, each a_n is not an accumulation point of E. In other words, for each $n \in \mathbb{N}$, there exist neighborhoods U_n of a_n that are relatively compact in U with disjoint closures. The proceeding proof is analogous to that of the existence part in Theorem 4.4.3.3.

Finally, we will examine the construction of entire functions interpolating prescribed values and derivatives at given points.

Let $\{z_j\}_{j=1}^{n} \subset \mathbb{C}$ be a sequence of distinct complex numbers and let $\{w_j\}_{j=1}^{n} \subset \mathbb{C}$ be a sequence of complex numbers. We can then construct a polynomial $f(z)$ such that $\forall j \in \{1,...,n\}$, $f(z_j) = w_j$. One such explicit formula is given by the *Lagrange interpolation formula*:

$$f(z) = \sum_{j=1}^{n}\left[w_j \prod_{\substack{k=1\\k\neq j}}^{n} \frac{z - z_k}{z_j - z_k}\right].$$

Then, following the assumption that $\{z_j\}_{j=1}^{n} \subset \mathbb{C}$ is a sequence of distinct complex numbers, let $\{w_{j,k}\}_{j\in\{1,...,n\},k\in\{0,...,n_j\}}$ be a sequence where

$\{n_j\}_{j=1}^{n} \subset \mathbb{N}$. Then we can find a polynomial $f(z)$ such that $\forall j \in \{1, ..., n\}$, $\forall k \in \{0, ..., n_j\}$, $f^{(k)}(z_j) = k! w_{j,k}$ (for clarity's sake, j selects the pair and k selects the order of the derivative, whose upper bound varies for each j). Oftentimes, the factorial coefficient is absorbed into $\{w_{j,k}\}$.

As it turns out, an entire function can in fact be constructed for infinitely many interpolation points, or when $n \to \infty$.

Theorem 4.4.3.5: Let $\{z_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ be a discrete set and let $\{w_{k,n}\}_{k\in\mathbb{N}, n\in\{0,...,n_k\}}$ be a sequence where $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. Then there exists an entire function such that $\forall k \in \mathbb{N}$, $\forall n \in \{0, ..., n_k\}$,

$$f^{(n)}(z_k) = n! w_{k,n}. \tag{4.4.10}$$

In other words, an entire function can be constructed by the given first n_k coefficients of the Taylor expansion at each z_k.

Proof: According to the Weierstrass Product Theorem (Theorem 4.4.3.1), we can construct an entire function $\Phi(z)$ with zeros at each of $\{z_k\}_{k\in\mathbb{N}}$ with corresponding multiplicities $\{n_k\}_{k\in\mathbb{N}}$. By the discreteness of $\{z_k\}_{k\in\mathbb{N}}$, there exists a corresponding sequence of radii $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that each $\overline{D(z_k, 2\varepsilon_k)}$ is disjoint.

Define a complex function sequence $\{\phi_k(z)\}_{k\in\mathbb{N}}$ by

$$\phi_k(z) = \sum_{n=0}^{n_k-1} w_{k,n}(z - z_k)^n,$$

where $k \in \mathbb{N}$. By Theorem 3.2.1.6, we can construct a C^∞ sequence of functions $\{\varphi_k(z)\}_{k\in\mathbb{N}}$ such that $\forall k \in \mathbb{N}$, $\operatorname{supp}(\varphi_k) \subset D(z_k, 2\varepsilon_k)$, $\varphi_k \equiv 1$ on $\overline{D(z_k, \varepsilon_k)}$, and $0 \leq \varphi_k \leq 1$ on $\mathbb{C}$.

Let $\Psi \in C^\infty(\mathbb{C})$, and construct

$$f(z) = -\Phi(z)\Psi(z) + \sum_{k=1}^{\infty} \phi_k(z)\varphi_k(z). \tag{4.4.11}$$

Under what conditions on Ψ will f be entire? Since the supports of each φ_k are disjoint, the summation $\sum_{k=1}^{\infty} \phi_k(z)\varphi_k(z)$ contains at most one nonzero term and is convergent and well-defined. To construct f to be entire, we must have $\frac{\partial f}{\partial \overline{z}} = 0$. In other words, we want

$$\frac{\partial}{\partial \overline{z}}\left(\sum_{k=1}^{\infty} \phi_k \varphi_k\right) = \frac{\partial}{\partial \overline{z}}(\Phi\Psi) \Longleftrightarrow \sum_{k=1}^{\infty} \phi_k \frac{\partial \varphi_k}{\partial \overline{z}} = \Phi \frac{\partial \Psi}{\partial \overline{z}}$$

on all of $\mathbb{C}$. Let

$$g(z) = \sum_{k=1}^{\infty} \phi_k(z) \frac{\partial \varphi_k(z)}{\partial \overline{z}}.$$

Since $\varphi_k \equiv 1$ on $\overline{D(z_k, \varepsilon_k)}$, $\frac{\partial \varphi_k}{\partial \overline{z}} \equiv 0$ on $\bigcup_{k=1}^{\infty} \overline{D(z_k, \varepsilon_k)}$. Consequently, $g(z) \equiv 0$ on $\bigcup_{k=1}^{\infty} \overline{D(z_k, \varepsilon_k)}$.

From rearrangement, we have

$$\frac{g(z)}{\Phi(z)} = \frac{\partial \Psi}{\partial \overline{z}},$$

which has removable singularities at each z_k. Define $\frac{g(z)}{\Phi(z)} = 0$ at $z = z_k$. Under this assertion, we have $\frac{g(z)}{\Phi(z)} \in C^{\infty}(\mathbb{C})$. Since the support of $\frac{g(z)}{\Phi(z)}$ is the union of disjoint compact sets, by Theorem 3.1.6, there exists a function $\Psi \in C^{\infty}(\mathbb{C})$ satisfying

$$\frac{g(z)}{\Phi(z)} = \frac{\partial \Psi}{\partial \overline{z}}.$$

Since g vanishes on $\bigcup_{k=1}^{\infty} \overline{D(z_k, \varepsilon_k)}$, it follows that $\frac{g}{\Phi}$ vanishes on $\bigcup_{k=1}^{\infty} \overline{D(z_k, \varepsilon_k)}$, and Ψ is holomorphic on $\bigcup_{k=1}^{\infty} D(z_k, \varepsilon_k)$.

Fix $k \in \mathbb{N}$ and let $n \in \{0, \ldots, n_k - 1\}$. For $z \in D(z_k, \varepsilon_k)$, from (4.4.11), we have

$$f(z) = -\Phi(z)\Psi(z) + \phi_k(z).$$

Since Φ has a zero at z_k with multiplicity n_k, $\Phi(z)\Psi(z)$ vanishes at z_k with multiplicity at least n_k. Therefore, we have

$$\begin{aligned} f^{(n)}(z_k) &= \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(\sum_{j=0}^{n_k - 1} w_{k,j} (z - z_k)^j \right) \Bigg|_{z_k} - \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(\sum_{j=n_k}^{\infty} w'_{k,j} (z - z_k)^j \right) \Bigg|_{z_k} \\ &= \lim_{z \to z_k} \sum_{j=n}^{n_k - 1} \left(w_{k,j} (z - z_k)^{j-n} \prod_{r=j-n+1}^{j} r \right) \\ &\quad - \sum_{j=n_k}^{\infty} \left(w'_{k,j} (z - z_k)^{j-n} \prod_{r=j-n+1}^{j} r \right) \\ &= n! w_{k,n}, \end{aligned}$$

as desired. □

Remark: For a general power series, there is no assurance that it corresponds to the Taylor expansion of an entire function. However, for any polynomial of degree n, there always exists an entire function whose Taylor expansion agrees with the polynomial up to the first $n + 1$ terms, which is the fundamental difference between a polynomial and a transcendental entire function.

Example 4.4.3.1: Prove the pole expansion formula

$$\pi \csc(\pi z) = \text{P.V.}\left(\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{z+k}\right) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z(-1)^k}{z^2 - k^2}$$

for $z \in \mathbb{C} \setminus \mathbb{Z}$.

Proof: Let the simple poles of $\pi \csc(\pi z)$ at each integer be enumerated by

$$a_n = \begin{cases} -\frac{n}{2} & \text{if } n \in 2\mathbb{N}, \\ \frac{n+1}{2} & \text{if } n \in \mathbb{N} \setminus 2\mathbb{N}. \end{cases}$$

□

4.4.4 Classifying Growth of Entire Functions

Lemma 4.4.4.1: Let $f : \overline{D(0,r)} \to \mathbb{C}^*$ (where $r > 0$) be a nowhere-vanishing holomorphic function. It follows that

$$\log|f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| \mathrm{d}\theta.$$

Proof: Without loss of generality, assume $r = 1$. Since f is non-vanishing and $\overline{\mathbb{D}}$ is simply connected, we may define the *holomorphic logarithm* as

$$\log(f(z)) = \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z + \log(f(z_0))$$

for any fixed $z_0 \in \overline{\mathbb{D}}$ and all $z \in \overline{\mathbb{D}}$, where $\gamma \subset \overline{\mathbb{D}}$ is any piecewise smooth curve from z_0 to z.

Hence, $\log|f(z)| = \mathfrak{Re}[\log(f(z))]$ and is therefore harmonic. The assertion then follows from the mean-value property. □

Theorem 4.4.4.1 *(Jensen's Formula)*: Let $f : \overline{D(0,r)} \to \mathbb{C}$ be meromorphic such that $f(0) \neq 0$. If $a_1, ..., a_m$, $b_1, ..., b_n$ are the zeros and poles of f in $\overline{D(0,r)}$, counted with multiplicities and orders, respectively, then

$$\log|f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| \mathrm{d}\theta + \sum_{k=1}^{m} \log\left|\frac{a_k}{r}\right| - \sum_{k=1}^{n} \log\left|\frac{b_k}{r}\right|.$$

Proof: For simplicity, assume $a_1, ..., a_{m_0}$ are the zeros in $D(0,r)$ and $a_{m_0+1}, ..., a_m$ are the zeros on $\partial D(0,r)$. Similarly, let $b_1, ..., b_{n_0}$ be the poles in $D(0,r)$ and $b_{n_0+1}, ..., b_n$ be the poles on $\partial D(0,r)$. Let

$$f(z) = g(z)\frac{\prod_{j=1}^m (z - a_j)}{\prod_{k=1}^n (z - b_k)},$$

where g is holomorphic and non-vanishing on $\overline{D(0,r)}$. Since

$$\log|f(0)| = \log|g(0)| + \sum_{j=1}^m \log|a_j| - \sum_{k=1}^n \log|b_k|,$$

by Lemma 4.4.4.1 on g,

$$\begin{aligned}
\log|f(0)| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\frac{\prod_{k=1}^n\left(re^{i\theta} - b_k\right)}{\prod_{j=1}^m\left(re^{i\theta} - a_j\right)}\right| \,d\theta \\
&\quad + \sum_{j=1}^m \log|a_j| - \sum_{k=1}^n \log|b_k| \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta + \sum_{j=1}^m \log|a_j| - \sum_{k=1}^n \log|b_k| \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{k=1}^n\left(\log r + \log\left|1 - \frac{b_k}{re^{i\theta}}\right|\right)\right. \\
&\qquad \left. - \sum_{j=1}^m\left(\log r + \log\left|1 - \frac{a_j}{re^{i\theta}}\right|\right)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta + \sum_{j=1}^m \log\left|\frac{a_j}{r}\right| - \sum_{k=1}^n \log\left|\frac{b_k}{r}\right| \\
&\quad + \mathfrak{Re}\,\frac{1}{2\pi i}\left[\left(\sum_{j=1}^m \oint_{\partial D(0,|a_j/r|)} - \sum_{k=1}^n \oint_{\partial D(0,|b_k/r|)}\right)\frac{\operatorname{Log}(1-z)\,dz}{z}\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta + \sum_{j=1}^m \log\left|\frac{a_j}{r}\right| - \sum_{k=1}^n \log\left|\frac{b_k}{r}\right| \\
&\quad + \mathfrak{Re}\,\frac{1}{2\pi i}\left(\sum_{k=n_0+1}^n - \sum_{j=m_0+1}^m\right)\oint_{\partial\mathbb{D}} \frac{\operatorname{Log}(1-z)\,dz}{z}
\end{aligned}$$

where $z = a_j/\left(re^{i\theta}\right), b_k/\left(re^{i\theta}\right)$, $d\theta = i\,dz/z$, and the leftover integrals (up until $k = n_0$ and $j = m_0$) for interior points vanish by Cauchy–Goursat (Theorem 3.1.7), since $\frac{\operatorname{Log}(1-z)}{z}$ has a removable singularity at $z = 0$.

We now are left to prove that the remaining integral I vanishes as well, which is not as immediate since the integrand does not extend continuously to the boundary. Let $z = \mathrm{e}^{\mathrm{i}\theta}$, $\mathrm{d}z = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}$, then (by $\psi = \frac{\theta}{2}$)

$$\begin{aligned}
I &= \oint_{\partial\mathbb{D}} \frac{\operatorname{Log}(1-z)\,\mathrm{d}z}{z} = \int_0^{2\pi} \log\left|1-\mathrm{e}^{\mathrm{i}\theta}\right| \mathrm{d}\theta = 2\int_0^{\pi} \log\left|\mathrm{e}^{-\mathrm{i}\psi} - \mathrm{e}^{\mathrm{i}\psi}\right| \mathrm{d}\psi \\
&= 2\pi \log 2 + 2\int_0^{\pi} \log|\sin\psi|\,\mathrm{d}\psi = 2\pi\log 2 + 4\int_0^{\pi/2} \log|\sin\psi|\,\mathrm{d}\psi \\
&= 2\pi\log 2 + 4J.
\end{aligned}$$

Splitting at $\pi/4$ and using cos with a substitution for the second integral then yields

$$J = \int_0^{\pi/4} \log|\sin\psi|\,\mathrm{d}\psi + \int_0^{\pi/4} \log|\cos\psi|\,\mathrm{d}\psi = \int_0^{\pi/4} \log\left|\frac{1}{2}\sin 2\psi\right| \mathrm{d}\psi.$$

Changing back to $\theta = 2\psi$, we have

$$J = \frac{J}{2} - \frac{\pi}{4}\log 2 \Longrightarrow 4J = -2\pi\log 2 \Longrightarrow I = 0. \qquad \square$$

As an immediate consequence, we have:

Corollary 4.4.4.1.1 *(Jensen's Inequality)*: Let f be holomorphic on $\overline{D(0,r)}$ such that $f \not\equiv 0$ and $f(0) \neq 0$. It follows that

$$\log|f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right| \mathrm{d}\theta.$$

Theorem 4.4.4.2 *(Poisson–Jensen Formula)*: Suppose f is a meromorphic function on $\overline{D(0,r)}$ such that $f \not\equiv 0$ on $D(0,r)$ and is non-vanishing and non-infinity on $\partial D(0,r)$. Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ be the zeros and poles of f in $D(0,r)$, counted with multiplicity and order, respectively (multiplicities and orders count as multiple zeros or poles). Then it follows that

$$\begin{aligned}
\log|f(z)| = &\int_0^{2\pi} \log|f(\zeta)| P(\zeta,z)\,\mathrm{d}\theta \\
&+ \sum_{j=1}^{m} \log\left|\frac{r\left(z-a_j\right)}{r^2 - \overline{a_j}z}\right| - \sum_{k=1}^{n} \log\left|\frac{r\left(z-b_k\right)}{r^2-\overline{b_k}z}\right|,
\end{aligned} \tag{4.4.12}$$

where $\zeta = r\mathrm{e}^{\mathrm{i}\theta}$, $z \in D(0,r) \setminus \left(\left\{a_j\right\}_{j=1}^{m} \cup \left\{b_k\right\}_{j=1}^{n}\right)$, and $P(\zeta,z)$ is the Poisson kernel in (4.4.6).

Proof: For fixed $z \in D(0,r)$ not at zeros or poles, let

$$g_z(\zeta) = f\left(r\varphi_{-\frac{z}{r}}\left(\frac{\zeta}{r}\right)\right)$$

where $\varphi_{-a} = (\varphi_a)^{-1}$ is the unit disk automorphism sending 0 to a. Then g_z maps 0 to $f(z)$, and has zeros at $r\varphi_{-\frac{z}{r}}\left(\frac{\zeta}{r}\right) = a_k$ or $\zeta = r\varphi_{\frac{z}{r}}\left(\frac{a_k}{r}\right)$ and poles at $r\varphi_{-\frac{z}{r}}\left(\frac{\zeta}{r}\right) = b_k$ or $\zeta = r\varphi_{\frac{z}{r}}\left(\frac{b_k}{r}\right)$. By Jensen's formula (Theorem 4.4.4.1),

$$\begin{aligned}\log|g_z(0)| = \log|f(z)| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|g_z\left(re^{i\theta}\right)\right| \mathrm{d}\theta \\ &+ \sum_{k=1}^{m}\left|\varphi_{\frac{z}{r}}\left(\frac{a_k}{r}\right)\right| - \sum_{k=1}^{n}\left|\varphi_{\frac{z}{r}}\left(\frac{b_k}{r}\right)\right|.\end{aligned}$$

□

Lemma 4.4.4.2: Let $f : \mathbb{D} \to \mathbb{C}$ be a non-constant bounded holomorphic function whose zeros are $a_1, a_2, \ldots$, counted according to their multiplicities, ordered such that $|a_n| \leq |a_{n+1}|$ for all $n \in \mathbb{N}$. Then,

$$\sum_{n=1}^{\infty}(1 - |a_n|)$$

is convergent.

Proof: First assume $f(0) \neq 0$ and choose M such that $|f| \leq M$ on $\mathbb{D}$. Let $n(r, 0, f)$ count the number of zeros of f, according to multiplicities, inside $\overline{D(0, r)}$. By Jensen's Formula (Theorem 4.4.4.1), we have

$$\begin{aligned}\log|f(0)| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| \mathrm{d}\theta \\ &+ \sum_{k=1}^{n(r,0,f)} \log\left|\frac{a_k}{r}\right| \leq \log(M) + \sum_{k=1}^{n(r,0,f)} \log\left|\frac{a_k}{r}\right|.\end{aligned}$$

For any fixed positive integer k, choose r such that $|a_k| < r < 1$. Then $n(r, 0, f) \geq k$ and

$$\sum_{k=1}^{n(r,0,f)} \log\left|\frac{a_j}{r}\right| \leq \sum_{j=1}^{k} \log\left|\frac{a_j}{r}\right|,$$

since each $\log\left|\frac{a_j}{r}\right| < 0$ for $j = k + 1, \ldots, n(r, 0, f)$. Therefore,

$$\log|f(0)| \leq \log M + \sum_{j=1}^{k} \log\left|\frac{a_j}{r}\right| = \log M + \sum_{j=1}^{k} \log|a_j| - k \log r.$$

Rearranging,

$$\sum_{j=1}^{k} \log\left|a_j\right| \geq \log|f(0)| - \log M + k \log r.$$

Now let $r \to 1^-$ with $r > |a_k|$. Since $k \log r \to 0$, it follows that

$$\sum_{j=1}^{k} \log\left|a_j\right| \geq \log|f(0)| - \log M.$$

This holds for every k. Since $\log\left|a_j\right| < 0$ for all j, the partial sums $\sum_{j=1}^{k} \log\left|a_j\right|$ are decreasing and bounded below by $\log|f(0)| - \log M$, hence converge to some finite limit, and

$$\sum_{j=1}^{\infty} \log\left|a_j\right| \geq \log|f(0)| - \log M,$$

or equivalently,

$$\log|f(0)| \leq \log|M| + \sum_{k=1}^{\infty} \log|a_k|.$$

For any $0 < a < 1$, we have $-\log(a) = 1 - a + \sum_{n=2}^{\infty} (1-a)^n a^n > 1 - a$. Hence,

$$0 \leq \sum_{k=1}^{\infty} (1 - |a_k|) < -\sum_{k=1}^{\infty} \log|a_k| \leq \log|M| - \log|f(0)|.$$

If f has a zero of multiplicity m at 0, then the argument applies to $z \mapsto \frac{f(z)}{z^m}$.□

Theorem 4.4.4.3 *(Blaschke Product)*: Let $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{D}^* = \mathbb{D} \setminus \{0\}$ be a sequence such that the series $\sum_{k=1}^{\infty}(1 - |a_k|)$ is convergent (known as the *Blaschke condition*). Then the *Blaschke product*, defined by

$$B(z) = \prod_{k=1}^{\infty} \left[-\frac{|a_k|}{a_k} \varphi_{a_k}(z) \right], \tag{4.4.13}$$

(where $\varphi_a(z)$ is a Möbius transformation in the form of (4.4.1)), locally uniformly converges to an analytic function on $\mathbb{D}$ such that $|B| \leq 1$ on $\mathbb{D}$, and its only zeros are precisely at each of $\{a_k\}_{k\in\mathbb{N}}$, counted according to multiplicities.

Proof: If it can be shown that

$$\sum_{k=1}^{\infty} \left| \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z} - 1 \right|$$

locally uniformly converges, we can use Lemma 4.4.3.3 to show that the infinite product converges uniformly on compact subsets of $\mathbb{D}$. Let $\overline{D(0,r)} \subset \mathbb{D}$ be a compact subset. The summand can be bounded with

$$\begin{aligned}\left|\frac{|a_k|}{a_k}\frac{a_k - z}{1 - \overline{a_k}z} - 1\right| &= \left|\frac{\overline{a_k}}{|a_k|}\frac{a_k - z}{1 - \overline{a_k}z} - 1\right| = \left|\frac{|a_k|^2 - \overline{a_k}z}{|a_k|(1 - \overline{a_k}z)} - 1\right| \\ &= \left|\frac{|a_k|^2 - \overline{a_k}z - |a_k| + |a_k|\overline{a_k}z}{|a_k|(1 - \overline{a_k}z)}\right| \\ &= \left|\frac{\overline{a_k}z(|a_k| - 1) + |a_k|(|a_k| - 1)}{|a_k|(1 - \overline{a_k}z)}\right| \\ &= \left|\frac{(\overline{a_k}z + |a_k|)(1 - |a_k|)}{|a_k|(1 - \overline{a_k}z)}\right| \\ &\leq (1 - |a_k|)\frac{|\overline{a_k}|(1 + r)}{|a_k|(1 - |a_k|r)} < (1 - |a_k|)\frac{1 + r}{1 - r}.\end{aligned}$$

Since

$$\sum_{k=1}^{\infty}\left|\frac{|a_k|}{a_k}\frac{a_k - z}{1 - \overline{a_k}z} - 1\right| < \frac{1 + r}{1 - r}\sum_{k=1}^{\infty}(1 - |a_k|)$$

is convergent (Blaschke condition), by the Weierstrass M-Test (Theorem 2.3.2), $\sum_{k=1}^{\infty}\left|\frac{|a_k|}{a_k}\frac{a_k - z}{1 - \overline{a_k}z} - 1\right|$ converges uniformly on $\overline{D(0,r)}$. By Lemma 4.4.3.3, the infinite product in (4.4.13) converges uniformly on compact subsets of $\mathbb{D}$. The properties of its zeros follow from the lemma.

Lastly, since $\left|\varphi_{a_k}\right| \leq 1$ and each partial product is bounded by 1, it follows that $|B(z)| \leq 1$ on $\mathbb{D}$. □

Remark: A more general Blaschke product has an additional factor of z^m to account for a zero at the origin, similar to the case of the Weierstrass product.

Corollary 4.4.4.3.1: Let $f : \mathbb{D} \to \mathbb{C}$ be bounded and holomorphic whose multiplicity of the zero at 0 is m (if f does not vanish at 0, then $m = 0$). If $\{a_n\}_{n\in\mathbb{N}}$ are its zeros in $\mathbb{D}^*$, counting multiplicities, then

$$f(z) = F(z)z^m\prod_{n=1}^{\infty}\left[-\frac{|a_n|}{a_n}\varphi_{a_n}(z)\right],$$

where F is bounded, holomorphic, and non-vanishing on $\mathbb{D}$. Moreover,

$$\sup_{z\in\mathbb{D}}|f(z)| = \sup_{z\in\mathbb{D}}|F(z)|.$$

Proof: Let

$$F(z) = \frac{f(z)}{z^m \prod_{n=1}^{\infty} \left[-\frac{|a_k|}{a_k} \varphi_{a_n}(z)\right]}.$$

By construction, F extends to its removable singularities to a holomorphic function that does not vanish. Because

$$\sup_{z \in \mathbb{D}} \left| z^m \prod_{n=1}^{\infty} \left[-\frac{|a_n|}{a_n} \varphi_{a_n}(z)\right] \right| \leq 1,$$

it follows that

$$\sup_{z \in \mathbb{D}} |F(z)| \geq \sup_{z \in \mathbb{D}} |f(z)|. \tag{4.4.14}$$

The partial products

$$B_n(z) = \prod_{k=1}^{n} \left[-\frac{|a_k|}{a_k} \varphi_{a_k}(z)\right]$$

give for fixed $\theta \in \mathbb{R}$, $\varepsilon > 0$, the existence of $0 < r' < 1$ such that $r' < r < 1$ implies

$$\left|B_n\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right| > 1 - \varepsilon.$$

Then by the Maximum Modulus Principle (Theorem 3.4.1),

$$\begin{aligned} \sup_{z \in \mathbb{D}} \left| \frac{f(z)}{z^m B_n(z)} \right| &= \sup_{z \in \mathbb{D} \setminus \overline{D(0,r)}} \left| \frac{f(z)}{z^m B_n(z)} \right| \\ &\leq \frac{1}{r^m (1 - \varepsilon)} \sup_{z \in \mathbb{D}} |f(z)| \to \frac{1}{1 - \varepsilon} \sup_{z \in \mathbb{D}} |f(z)| \end{aligned}$$

as $r \to 1^-$. Letting $\varepsilon \to 0^+$, $n \to \infty$ gives

$$\sup_{z \in \mathbb{D}} |F(z)| \leq \sup_{z \in \mathbb{D}} |f(z)|,$$

which in conjunction with (4.4.14), completes the final assertion. □

From the results above, a recurring theme in complex analysis is hinted at; the rate of growth of functions provides insight towards the distribution of its zeros.

The subjects to be discussed here are relevant and preliminary to Nevanlinna theory, or the study of holomorphic value distribution.

For an entire function f, let $M(r, f) = \sup_{|z|=r} |f(z)| = \sup_{|z| \leq r} |f(z)|$ (by the Maximum Modulus Principle in Theorem 3.4.1).

Definition 4.4.4.1 *(Growth Order of Entire Functions)*: An entire function f is said to be of *finite order* if there exists $\alpha, r_\alpha \in \mathbb{R}$ such that

$$M(r, f) \leq \exp(r^\alpha), \quad \forall r > r_\alpha,$$

or in loose terms, f is of finite order if it grows at most exponentially for large z. The *order* of f, or $\rho(f)$ is defined to be the infimum of all α satisfying the previous condition.

Proposition 4.4.4.1: Let f be entire; then if there exist $a, b, \alpha, r_{\alpha,\beta} > 0$ such that

$$M(r, f) \leq \exp(ar^\alpha + b), \quad \forall r > r_{\alpha,\beta},$$

then $\rho(f) \leq \alpha$.

Proof: For $\varepsilon > 0$, since $r^\varepsilon \to \infty$ as $r \to \infty$, for any $\varepsilon > 0$, there exists r_ε such that

$$r^\varepsilon \geq 2a \Longrightarrow \frac{1}{2} r^{\alpha+\varepsilon} \geq ar^\alpha$$

for $r > r_\varepsilon$. There exists $r'_\varepsilon > 0$ such that

$$r > r'_\varepsilon \Longrightarrow \frac{1}{2} r^{\alpha+\varepsilon} \geq b.$$

For simplicity, let the value $\max\{r_\varepsilon, r'_\varepsilon\}$ be denoted by r_ε. Then

$$r > r_\varepsilon \Longrightarrow ar^\alpha + b \leq \frac{1}{2} r^{\alpha+\varepsilon} + \frac{1}{2} r^{\alpha+\varepsilon} = r^{\alpha+\varepsilon}.$$

By assumption, we have

$$M(r, f) \leq \exp(ar^\alpha + b) \leq \exp(r^{\alpha+\varepsilon}) \Longrightarrow \alpha + \varepsilon \geq \rho(f).$$

Letting $\varepsilon \to 0^+$, the assertion follows. □

Theorem 4.4.4.4: The order of an entire f may be explicitly given by

$$\rho(f) = \limsup_{r \to \infty} \frac{\log(\log M(r, f))}{\log r}.$$

Proof: By assumption, we have $\forall \varepsilon' > 0, \exists 0 < \varepsilon < \varepsilon'$ (or simply just $\forall \varepsilon > 0$ by the nature of the exponential) such that

$$M(r, f) \leq \exp\left(r^{\rho(f)+\varepsilon}\right)$$

for some r' and any $r > r'$. Taking logarithms twice we have

$$\frac{\log(\log M(r,f))}{\log r} \leq \limsup_{r\to\infty} \frac{\log(\log M(r,f))}{\log r} \leq \rho(f) + \varepsilon \to \rho(f)$$

as $\varepsilon' \to 0$. Moreover, for any $\varepsilon > 0$, $r' > 0$, $\exists r > r'$ such that

$$M(r,f) > \exp\left(r^{\rho(f)-\varepsilon}\right) \Longrightarrow \limsup_{r\to\infty} \frac{\log(\log M(r,f))}{\log r} \geq \rho(f) - \varepsilon \to \rho(f)$$

as $\varepsilon \to 0$. Therefore,

$$\rho(f) \leq \limsup_{r\to\infty} \frac{\log(\log M(r,f))}{\log r} \leq \rho(f). \qquad \square$$

Example 4.4.4.1: The function sin is of order 1, while exp ∘ exp is not of finite order.

Proof: We consider the two examples separately:

1 Observe that

$$\sup_{|z|=r} |\sin(z)| \leq \sup_{|z|=r} \frac{\left|\mathrm{e}^{\mathrm{i}z}\right| + \left|\mathrm{e}^{-\mathrm{i}z}\right|}{2} = \sup_{|z|=r} \frac{\mathrm{e}^{|y|} + \mathrm{e}^{-|y|}}{2} \leq \sup_{|z|=r} \mathrm{e}^{|y|} = \mathrm{e}^r.$$

For $r > 1$, we have $\mathrm{e}^{-r} < 1 < \frac{1}{2}\mathrm{e}^r$, and hence for $z = \mathrm{i}r$, we have

$$|\sin(z)| = \frac{\mathrm{e}^r - \mathrm{e}^{-r}}{2} > \frac{1}{4}\mathrm{e}^r.$$

Therefore,

$$\frac{1}{4}\mathrm{e}^r < \sup_{|z|=r} |\sin(z)| \leq \mathrm{e}^r \Longrightarrow \rho(f) = \limsup_{r\to\infty} \frac{\log(r + \mathcal{O}(1))}{\log r} = 1.$$

2 Let $z = r$, then

$$\begin{aligned} \sup_{|z|=r} |\exp\circ\exp| \geq \exp(\exp(r)) &\Longrightarrow \log\circ\log \sup_{|z|=r} |f(z)| \geq r \\ &\Longrightarrow \rho(f) \geq \limsup_{r\to\infty} \frac{r}{\log r} = \infty. \square \end{aligned} \qquad \square$$

The utility of ρ is that it gives implications on the rate of which the zeros of an entire function tend to ∞. The order for meromorphic functions is more general and is pertinent in Nevanlinna Theory (@ sec:nevanlinnatheory). This is quantified technically by the convergence range of the sum given by

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}},$$

where each a_n is a zero. Specifically, the infimum of all such k under which the prescribed sum converges correlates to this right. For example, let $a_n = n$ for each n. Then for any $k > 0$, the integral test gives the convergence of the series, while if $a_n = \sqrt{a_n}$ (corresponding to a slower approach to ∞), the series converges for $k > 1$.

For the following discussions, let $n(r, 0, f)$ count the zeros of f in $D(0, r)$ according to multiplicity.

Lemma 4.4.4.3: If f is entire with $f(0) = 1$, then

$$\log 2 \cdot n(r, 0, f) \leq \log M(2r, f).$$

Proof: By Jensen's formula (Theorem 4.4.4.1), for $r > 0$, we have

$$\sum_{k=1}^{n(2r,0,f)} \log\left|\frac{2r}{a_k}\right| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|f\left(2r\mathrm{e}^{\mathrm{i}\theta}\right)\right| \,\mathrm{d}\theta,$$

where $a_1, \dots, a_{n(2r,0,f)}$ are the zeros of f in $D(0, 2r)$, ordered such that each $|a_k| \leq |a_{k+1}|$. Then

$$\begin{aligned}\sum_{k=1}^{n(r,0,f)} \log 2 \leq \sum_{k=1}^{n(r,0,f)} \log\left|\frac{2r}{a_k}\right| &\leq \sum_{k=1}^{n(2r,0,f)} \log\left|\frac{2r}{a_k}\right| \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log\left|f\left(2r\mathrm{e}^{\mathrm{i}\theta}\right)\right| \,\mathrm{d}\theta \leq \log M(2r, f).\end{aligned}$$

☐

Theorem 4.4.4.5: For a nonzero complex sequence $\{a_k\}_{k\in\mathbb{N}}$ counting multiplicities (such that $|a_1| \leq |a_2|$, etc.), the sum

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{\sigma}}$$

converges for any

$$\sigma > \limsup_{r\to\infty} \frac{\log n(r)}{\log r}$$

where $n(r)$ counts a_k in the closed disk of radius r.

Proof: Choose σ' such that

$$\sigma > \sigma' > \limsup_{r\to\infty} \frac{\log n(r)}{\log r}.$$

For sufficiently large r,

$$\frac{\log n(r)}{\log r} < \sigma' \implies n(r) \leq r^{\sigma'}.$$

For sufficiently large $k \in \mathbb{N}$, by the ordering of zeros, it follows that

$$k \leq n(|a_k| + \delta) \leq (|a_k| + \delta)^{\sigma'}$$

for sufficiently small δ. As $\delta \to 0^+$, we have

$$k \leq |a_k|^{\sigma'} \implies \frac{1}{k} \geq \frac{1}{|a_k|^{\sigma'}} \implies \frac{1}{k^{\sigma/\sigma'}} \geq \frac{1}{|a_k|^{\sigma}}.$$

By the comparison test, we then have the convergence of

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{\sigma}}.$$

□

Theorem 4.4.4.6: For an entire function f ($f(0) = 1$) of finite order $\rho(f)$ whose zeros are at $\{a_k\}_{k\in\mathbb{N}}$ counting multiplicities (such that $|a_1| \leq |a_2|$, etc.), the sum

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{\rho(f)+\eta}}$$

converges for any $\eta > 0$.

Proof: By trivial definition, we have

$$M(2r, f) \leq \exp((2r)^{\rho+\varepsilon})$$

for all $\varepsilon' > 0$ and some $0 < \varepsilon < \varepsilon'$. Lemma 4.4.4.3 gives that for any $r > 0$,

$$\log 2 \cdot n(r, 0, f) \leq \log M(2r, f).$$

Hence,

$$\log 2 \cdot n(r, 0, f) \leq (2r)^{\rho(f)+\varepsilon} \iff \frac{n(r, 0, f)}{r^{\rho(f)+2\varepsilon}} \leq \frac{1}{\log 2} 2^{\rho(f)+\varepsilon} r^{-\varepsilon} \to 0^+$$

as $r \to \infty$. Then for sufficiently large r, we have

$$n \leq \frac{n(r, 0, f)}{r^{\rho(f)+2\varepsilon}} \leq 1 \implies n(r, 0, f) \leq r^{\rho(f)+2\varepsilon}.$$

For sufficiently large $k \in \mathbb{N}$, by the ordering of zeros, it follows that

$$k \leq n(|a_k| + \delta, 0, f) \leq (|a_k| + \delta)^{\rho(f)+2\varepsilon}$$

for sufficiently small δ. As $\delta \to 0^+$, we have

$$k \leq |a_k|^{\rho(f)+2\varepsilon} \Longrightarrow \frac{1}{k} \geq \frac{1}{|a_k|^{\rho(f)+2\varepsilon}} \Longrightarrow \frac{1}{k^{(\rho(f)+\eta)/(\rho(f)+2\varepsilon)}} \geq \frac{1}{|a_k|^{\rho(f)+\eta}}.$$

The left-hand side as a summation is convergent for $2\varepsilon < \eta$ or lower, and hence we have the convergence of

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{\rho(f)+\eta}}. \qquad \square$$

Therefore, for any $r > 0$, the series

$$\sum_{k=1}^{\infty} \left|\frac{r}{a_k}\right|^{\rho(f)+\eta} \leq \sum_{k=1}^{\infty} \left|\frac{r}{a_k}\right|^{\lfloor\rho\rfloor+1} \quad \text{for sufficiently small } \eta$$

converges. Then by the Weierstrass Factorization Theorem (Theorem 4.4.3.2),

$$f(z) = z^m \mathrm{e}^{\varphi(z)} \prod_{k=1}^{\infty} E_{\lfloor\rho\rfloor}\left(\frac{z}{a_k}\right)$$

locally uniformly converges on $\mathbb{C}$, where φ is entire.

Definition 4.4.4.2: The *rank* of an entire function is the smallest $p \in \mathbb{Z}_{\geq 0}$ for which the associated sum

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{p+1}}$$

converges, where $\{a_k\}_k$ are its zeros in $\mathbb{C}^*$.

The conclusion of Theorem 4.4.4.6 is that the rank of an entire function with finite order is finite. Moreover, the rank $\leq \lfloor\rho\rfloor$.

Definition 4.4.4.3: Let f be entire of finite rank p. By the Weierstrass Factorization theorem (Theorem 4.4.3.2),

$$f(z) = z^m \mathrm{e}^{\varphi(z)} \prod_{k=1}^{\infty} E_p\left(\frac{z}{a_k}\right).$$

If φ is a polynomial of degree q, then f is said to be of finite *genus* $\mu = \max\{p, q\}$.

This particular Weierstrass factorization is the *Weierstrass canonical factorization* of f (the portion corresponding to the product of elementary factors itself is the *Weierstrass canonical product*). Now that we have indulged in the implications of $\rho(f)$ to its zero distribution, we now turn to the function φ in the exponential.

Lemma 4.4.4.4: Let f be entire with finite order such that $f(0) = 1$. Let $\{a_k\}_{k\in\mathbb{N}}$ be the zeros of f, listed with multiplicities, such that $|a_1| \leq |a_2| \leq |a_3| \leq \ldots$. Suppose $p > \rho(f) - 1$; then for any $z \in \mathbb{C}$,

$$\lim_{r\to\infty} \sum_{k=1}^{n(r,0,f)} \overline{a_k}^{p+1} \left(r^2 - \overline{a_k} z\right)^{-p-1} = 0.$$

Proof: For fixed z, let $r > 2|z|$ such that $a_1, \ldots, a_{n(r,0,f)}$ lie in $D(0, r)$. For each k we obtain

$$\begin{aligned} \left|r^2 - \overline{a_k} z\right| &\geq r^2 - |a_k||z| > r^2 - r \cdot \frac{r}{2} = \frac{r^2}{2} \\ \Longrightarrow |a_k|^{p+1} \left|r^2 - \overline{a_k} z\right|^{-p-1} &< \left(\frac{2}{r}\right)^{p+1} \end{aligned}$$

since $\rho(f) \geq 0$ by the logarithm formula. Now by definition of $\rho(f)$, Lemma 4.4.4.3 gives the estimate for sufficiently large r and arbitrarily small $\varepsilon > 0$:

$$n(r, 0, f) r^{-p-1} \leq \frac{\log M(2r, 0, f)}{\log 2} r^{-p-1} \leq \frac{(2r)^{\rho(f)+\varepsilon} r^{-p-1}}{\log 2}.$$

Thus,

$$\left|\sum_{k=1}^{n(r,0,f)} \overline{a_k}^{p+1} \left(r^2 - \overline{a_k} z\right)^{-p-1}\right| \leq n(r, 0, f) \left(\frac{2}{r}\right)^{p+1} \leq \frac{r^{\rho(f)+\varepsilon-p-1} 2^{\rho(f)+\varepsilon+p+1}}{\log 2}.$$

Letting $\varepsilon = \frac{p+1-\rho(f)}{2}$ (positive by theorem assumption), we obtain

$$\left|\sum_{k=1}^{n(r,0,f)} \overline{a_k}^{p+1} \left(r^2 - \overline{a_k} z\right)^{-p-1}\right| \leq \frac{2^{(3p+3-\rho(f))/2} r^{-\varepsilon}}{\log 2} \to 0 \quad \text{as} \quad r \to \infty. \square$$

Theorem 4.4.4.7: Let f be entire with $f(0) = 1$. Then for $p > \rho(f) - 1$ (p integer) and $z \in \mathbb{C}$,

$$\lim_{r\to\infty} \int_0^{2\pi} \frac{r e^{i\theta} \log\left|f\left(r e^{i\theta}\right)\right|}{\left(r e^{i\theta} - z\right)^{p+2}} \, d\theta = 0.$$

Proof: For fixed z, $r > 2|z|$, we have

$$\int_0^{2\pi} \frac{i r e^{i\theta} \, d\theta}{\left(r e^{i\theta} - z\right)^{p+2}} = \oint_{\partial D(0,r)} \frac{dw}{(w - z)^{p+2}} = 2\pi i \operatorname*{Res}_{w=z} \frac{1}{(w - z)^{p+2}}$$

by the Residue Theorem (Theorem 4.5.1). Since $p+2 > \rho(f)+1 \geq 1$ where p is an integer, we must have $p+2 \geq 2$ and thus

$$\int_0^{2\pi} \frac{r\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{(r\mathrm{e}^{\mathrm{i}\theta}-z)^{p+2}} = 0.$$

Therefore,

$$\begin{aligned}
&\left|\int_0^{2\pi} \frac{r\mathrm{e}^{\mathrm{i}\theta}\log\left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|}{(r\mathrm{e}^{\mathrm{i}\theta}-z)^{p+2}}\,\mathrm{d}\theta\right| \\
&\qquad = \left|\int_0^{2\pi} \frac{r\mathrm{e}^{\mathrm{i}\theta}}{(r\mathrm{e}^{\mathrm{i}\theta}-z)^{p+2}}\left[\log\left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right| - \log M(r,f)\right]\mathrm{d}\theta\right| \\
&\qquad \leq \int_0^{2\pi} \frac{r}{(r/2)^{p+2}}\left[\log M(r,f) - \log\left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|\right]\mathrm{d}\theta \\
&\qquad = 2^{p+3}r^{-p-1}\left[2\pi\log M(r,f) - \int_0^{2\pi}\log\left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\theta\right] \\
&\qquad \leq 2^{p+4}r^{-p-1}\pi\log M(r,f),
\end{aligned}$$

where the last expression uses the inequality derived from Jensen's formula (Corollary 4.4.4.1.1) on the remaining integral.

Now by assumption, we have

$$\log M(r,f) \leq r^{\rho+\varepsilon}$$

for any $\varepsilon > 0$ and sufficiently large r. Hence,

$$\left|\int_0^{2\pi} \frac{r\mathrm{e}^{\mathrm{i}\theta}\log\left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|}{(r\mathrm{e}^{\mathrm{i}\theta}-z)^{p+2}}\,\mathrm{d}\theta\right| \leq 2^{p+4}r^{\rho(f)+\varepsilon-p-1}\pi = 2^{p+4}r^{(\rho(f)-p-1)/2}\pi$$

at $\varepsilon = \frac{p+1-\rho(f)}{2}$. Then since $\frac{\rho(f)-p-1}{2} < 0$, the expression vanishes as $R \to \infty$. ☐

Proposition 4.4.4.2: Let f be entire, non-constant, and of finite order such that $f(0)=1$. Let $\{a_k\}_{k\in\mathbb{N}}$ be the zeros of f counted according to multiplicities such that $|a_1| \leq |a_2| \leq |a_3| \leq \ldots$. If $p > \rho(f)-1$ is an integer, then

$$\frac{\mathrm{d}^p}{\mathrm{d}z^p}\left(\frac{f'(z)}{f(z)}\right) \equiv -\sum_{k=1}^{\infty}\frac{p!}{(a_k-z)^{p+1}}$$

for all $z \in \mathbb{C}$.

Proof: Let $r > 2|z|$. By the Poisson–Jensen Formula (Theorem 4.4.4.2), at each non-singular point, we have (the kernel representation derived in (4.4.6))

$$\begin{aligned}\mathfrak{Re}\log f(z) &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathfrak{Re}\left(\frac{r\mathrm{e}^{\mathrm{i}\theta}+z}{r\mathrm{e}^{\mathrm{i}\theta}-z}\right)\mathrm{d}\theta\\ &\quad+\sum_{k=1}^{n(r,0,f)}\mathfrak{Re}\log\left(\frac{r(z-a_k)}{r^2-\overline{a_k}z}\right).\end{aligned}$$

For any holomorphic $g = u + \mathrm{i}v$, we have

$$\begin{aligned}\frac{\partial(\mathfrak{Re}\,g(z))}{\partial z} &= \frac{1}{2}\left(\frac{\partial u(z)}{\partial x}-\mathrm{i}\frac{\partial u(z)}{\partial y}\right)\\ &= \frac{1}{2}\left(\frac{\partial u(z)}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}\right)=\frac{1}{2}\frac{\partial g(z)}{\partial x}=\frac{g'(z)}{2}.\end{aligned}\tag{4.4.15}$$

Therefore, by differentiation under the integral sign,

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{1}{\pi}\int_0^{2\pi}\log\left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|\frac{r\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{\left(r\mathrm{e}^{\mathrm{i}\theta}-z\right)^2}+\sum_{k=1}^{n(r,0,f)}\frac{r^2-|a_k|^2}{(r^2-\overline{a_k}z)(z-a_k)}\\ &= \frac{1}{\pi}\int_0^{2\pi}\log\left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|\frac{r\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{\left(r\mathrm{e}^{\mathrm{i}\theta}-z\right)^2}+\sum_{k=1}^{n(r,0,f)}\frac{\overline{a_k}}{r^2-\overline{a_k}z}+\sum_{k=1}^{n(r,0,f)}\frac{1}{z-a_k}.\end{aligned}$$

Differentiating p times from here gives

$$\begin{aligned}\frac{\mathrm{d}^p}{\mathrm{d}z^p}\left(\frac{f'(z)}{f(z)}\right) &= \frac{1}{\pi}\int_0^{2\pi}\log\left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|\frac{r\mathrm{e}^{\mathrm{i}\theta}(p+1)!\,\mathrm{d}\theta}{\left(r\mathrm{e}^{\mathrm{i}\theta}-z\right)^{p+2}}\\ &\quad+\sum_{k=1}^{n(r,0,f)}\frac{\overline{a_k}^{p+1}p!}{\left(r^2-\overline{a_k}z\right)^{p+1}}-\sum_{k=1}^{n(r,0,f)}\frac{p!}{\left(a_k-z\right)^{p+1}}.\end{aligned}$$

The first two terms vanish as $r \to \infty$ by Theorem 4.4.4.7 and Lemma 4.4.4.4.□

Lemma 4.4.4.5 *(Logarithmic Factorization)*: Let f be entire, non-constant, and of finite order ρ such that $f(0) = 1$. Let

$$P(z) = \prod_{k=1}^{\infty} E_{\operatorname{rank} f}\left(\frac{z}{a_k}\right)$$

be the associated product. If $p > \rho(f) - 1$ is an integer, then

$$\frac{\mathrm{d}^p}{\mathrm{d}z^p}\left(\frac{P'(z)}{P(z)}\right) \equiv -\sum_{k=1}^{\infty}\frac{p!}{\left(a_k-z\right)^{p+1}}$$

for all $z \in \mathbb{C}$.

Proof: Let P_n be the n-th partial product of P. Then

$$\frac{P_n'(z)}{P_n(z)} = \sum_{k=1}^{n} \frac{\frac{\mathrm{d}}{\mathrm{d}z} E_{\operatorname{rank} f}\left(\frac{z}{a_k}\right)}{E_{\operatorname{rank} f}\left(\frac{z}{a_k}\right)} = \sum_{k=1}^{n} \left[\frac{1}{z - a_k} + \sum_{j=1}^{\operatorname{rank} f} \frac{z^{j-1}}{a_k^j}\right],$$

implying that

$$\frac{\mathrm{d}^p}{\mathrm{d}z^p}\left(\frac{P_n'(z)}{P_n(z)}\right) = -\sum_{k=1}^{n} \frac{p!}{(a_k - z)^{p+1}} + \frac{\mathrm{d}^p}{\mathrm{d}z^p} \sum_{j=1}^{\operatorname{rank} f} \sum_{k=1}^{n} \frac{z^{j-1}}{a_k^j}.$$

Since the polynomial in the rightmost term has degree at most $\max j - 1 = \operatorname{rank} f - 1$, and because $p > \lfloor \rho \rfloor - 1 \geq \operatorname{rank} f - 1$, after p derivatives each term of the expression vanishes. For an arbitrarily chosen compact $K \subset \mathbb{C}$ avoiding a_k, some $N \in \mathbb{N}$ such that $|a_k| \geq \max_{z \in K} |z|$ for all $k > N$, we have $\forall z \in K$, $|a_k - z| \leq |a_k| + |z| \leq 2|a_k|$. Then the convergence of

$$\sum_{k=1}^{\infty} \frac{1}{2|a_k|^{p+1}}$$

from Theorem 4.4.4.6 implies the absolute convergence of

$$\sum_{k=1}^{\infty} \frac{1}{(a_k - z)^{p+1}}$$

in K. Moreover, it can be shown that the uniform convergence of $P_n \to P$ (from the Weierstrass factorization) and $P_n' \to P_n'$ (by the Weierstrass Convergence Theorem, Theorem 4.1.1) in K implies that of $\frac{P'(z)}{P(z)}$. Hence, the Weierstrass Convergence Theorem implies that

$$\begin{aligned} \lim_{n \to \infty} \frac{\mathrm{d}^p}{\mathrm{d}z^p}\left(\frac{P_n'(z)}{P_n(z)}\right) = \frac{P'(z)}{P(z)} &= -\lim_{n \to \infty} \sum_{k=1}^{n} \frac{p!}{(a_k - z)^{p+1}} \\ &= -\sum_{k=1}^{\infty} \frac{p!}{(a_k - z)^{p+1}}. \end{aligned}$$

□

The two preceding results are similar in conclusion, but Lemma 4.4.4.5 is not a special case of Proposition 4.4.4.2 since we have not asserted that the canonical product is of finite order.

4.4.5 Hadamard Factorization Theorem

Theorem 4.4.5.1: Let $f(z) = \mathrm{e}^{\varphi(z)}P(z)$ be the Weierstrass canonical factorization of f, where f is entire with finite order $\rho = \rho(f)$ and $\rho(0) = 1$. Then φ is a polynomial of degree $\leq \rho$.

Proof: By logarithmic differentiation and by taking $p > \rho - 1$ subsequent derivatives, we have

$$\frac{f'(z)}{f(z)} = \varphi'(z) + \frac{P'(z)}{P(z)} \Longrightarrow \frac{\mathrm{d}^p}{\mathrm{d}z^p}\left(\frac{f'(z)}{f(z)}\right) = \varphi^{(p+1)}(z) + \frac{\mathrm{d}^p}{\mathrm{d}z^p}\left(\frac{P'(z)}{P(z)}\right).$$

By applying Proposition 4.4.4.2 and Lemma 4.4.4.5, we have

$$-\sum_{k=1}^{\infty} \frac{p!}{(a_k - z)^{p+1}} = \varphi^{(p+1)}(z) - \sum_{k=1}^{\infty} \frac{p!}{(a_k - z)^{p+1}} \Longrightarrow \varphi^{(p+1)} \equiv 0.$$

Hence, φ is a polynomial of degree $\leq p$. Choosing $p = 1 + \lfloor \rho - 1 \rfloor > \rho - 1$ so that $p \leq \rho$, the assertion follows. □

Corollary 4.4.5.1.1: Let $f(z) = z^m \mathrm{e}^{\varphi(z)}P(z)$ be the Weierstrass canonical factorization of f, where f is entire with finite order $\rho = \rho(f)$. Then φ is a polynomial of degree $\leq \rho$.

Proof: Let $f(z) = z^m g(z)$, where f and g are entire, $g(0) \neq 0$, and f has finite order $\rho(f)$ and $M(r, f) = r^m M(r, g)$ for $r > 0$. For $\varepsilon > 0$, $\exists r' > 0$ such that $r > r'$ implies

$$M(r, f) = r^m M(r, g) \leq \mathrm{e}^{r^{\rho(f)+\varepsilon}} \Longrightarrow M(r, g) \leq \mathrm{e}^{r^{\rho(f)+\varepsilon}}.$$

Thus, $\rho(g) \leq \rho(f)$ by letting $\varepsilon \to 0^+$. Additionally, for any $\varepsilon > 0$, $\forall r' > 0$, $\exists r > r'$ such that

$$M(r, f) \geq \mathrm{e}^{r^{\rho(f)-\varepsilon}} \Longrightarrow M(r, g) \geq \exp\left(r^{\rho(f)-2\varepsilon}\left(r^{\varepsilon} - \frac{m \log r}{r^{\rho(f)-2\varepsilon}}\right)\right) \geq \mathrm{e}^{r^{\rho(f)-2\varepsilon}}$$

because for sufficiently large r,

$$r^{\varepsilon} - \frac{m \log r}{r^{\rho(f)-2\varepsilon}} > 1.$$

Hence, $\rho(g) \geq \rho(f) - 2\varepsilon$. Letting $\varepsilon \to 0^+$ implies $\rho(f) = \rho(g)$. Let $g(z) = ch(z)$ where c is a constant, so that $h(0) = 1$. It is also trivial that $\rho(g) = \rho(h)$. Explicitly, we have $h(z) = \mathrm{e}^{\varphi - \operatorname{Log} c}P(z)$.

By Theorem 4.4.5.1 on h, $\varphi - \operatorname{Log} c$ is a polynomial of degree $\leq \rho$, and so is φ. □

Then the results of Corollary 4.4.5.1.1 and Theorem 4.4.4.6 may be consolidated into a single statement:

Theorem 4.4.5.2 *(Hadamard Factorization Theorem)*: Let μ be the genus of f and let ρ be the order of f, where f is entire with finite order. Then $\mu \leq \rho$.

Theorem 4.4.5.3: The factorization

$$\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 k^2} \right)$$

defines an entire function and uniformly converges on any compact disk $\overline{D(0,r)}$.

Proof: The zeros of sin are simple at each of $\mathbb{Z}$. Aside from the simple zero at $z = 0$, let

$$a_k = \begin{cases} -\pi k/2 & \text{if } k \in 2\mathbb{N}, \\ \pi(k+1)/2 & \text{if } k \in \mathbb{N} \setminus 2\mathbb{N} \end{cases}$$

enumerate the zeros of sin. By Example 4.4.4.1, and the Hadamard Factorization Theorem (Theorem 4.4.5.2), the order of sin is 1, the genus does not exceed 1, and

$$\sin z = z e^{\varphi(z)} \prod_{k=1}^{\infty} E_1\left(\frac{z}{a_k}\right) = z e^{\varphi(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) \exp\left(\frac{z}{a_k}\right),$$

where $\varphi(z) = az + b$ is a polynomial (and where the product locally uniformly converges in $\mathbb{C}$). Since the partial products $\{P_n\}_{n \in \mathbb{N}}$, where

$$P_n = \prod_{k=1}^{n} \left(1 - \frac{z}{a_k}\right) \exp\left(\frac{z}{a_k}\right),$$

have a single accumulation point, the subsequence $\{P_{2n}\}_{n \in \mathbb{N}}$ converges to the same point. Since

$$\begin{aligned} P_{2n} &= \prod_{k=1}^{2n} \left(1 - \frac{z}{a_k}\right) \exp\left(\frac{z}{a_k}\right) \\ &= \prod_{k=1}^{n} \left[\left(1 - \frac{z}{\pi k}\right) \exp\left(\frac{z}{\pi k}\right) \left(1 + \frac{z}{\pi k}\right) \exp\left(-\frac{z}{\pi k}\right) \right], \end{aligned}$$

we have

$$\sin z = z e^{az+b} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 k^2}\right).$$

Then from $\sin z = -\sin(-z)$ we have

$$z\mathrm{e}^{az+b}\prod_{k=1}^{\infty}\left(1-\frac{z^2}{\pi^2 k^2}\right) \equiv z\mathrm{e}^{-az+b}\prod_{k=1}^{\infty}\left(1-\frac{z^2}{\pi^2 k^2}\right) \Longrightarrow \mathrm{e}^{2az} \equiv 1 \Longrightarrow a = 0.$$

Since $\lim_{\zeta\to 0}\frac{\sin\zeta}{\zeta} = 1$, we have

$$\lim_{z\to 0}\mathrm{e}^{b}\prod_{k=1}^{\infty}\left(1-\frac{z^2}{\pi^2 k^2}\right) = 1 \Longrightarrow b = 0. \qquad \square$$

4.5 The Residue Theorem

After Riemann and Weierstrass refined the understanding of analytic functions and the formal characterization of Jordan curves, the Cauchy Residue Theorem was consequently formalized. Cauchy had the informal notion of a residue, which we will now formally introduce.

Definition 4.5.1 *(Residue)*: For some $r \in \mathbb{R}_{>0}$, $a \in U$, suppose $f : D^*(a, r) \to \mathbb{C}$ is holomorphic. Then the *residue* of f at a, denoted by $\operatorname{Res}_{z=a} f(z)$ or $\operatorname{Res}(f, a)$, is equal to

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{2\pi\mathrm{i}}\oint_{\partial D(a,\rho)} f(z)\,\mathrm{d}z, \tag{4.5.1}$$

where $0 < \rho < r$ is arbitrary. Since f has a Laurent expansion at a, being

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n, \quad c_n = \frac{1}{2\pi\mathrm{i}}\oint_{\partial D(a,\rho)}\frac{f(z)\,\mathrm{d}z}{(z-a)^{n+1}},$$

we get that the residue of f at a is equal to the first term c_{-1} of the principal part of its Laurent expansion.

It then follows that the residue at a removable singularity is 0. As a direct consequence of (4.5.1), we can derive explicit formulas for the calculation of residues at poles. If $U \subseteq \mathbb{C}$ is open, $a \in U$ is an isolated singularity (a pole of order $m \neq \infty$) of $f : U \setminus \{a\} \to \mathbb{C}$ that is holomorphic, then locally:

$$f(z) = c_{-m}(z-a)^{-m} + c_{1-m}(z-a)^{1-m} + \cdots + c_{-1}(z-a)^{-1} + \ldots.$$

Multiplying by $(z-a)^m$, we obtain that

$$(z-a)^m f(z) = c_{-m} + c_{1-m}(z-a) + \cdots + c_{-1}(z-a)^{m-1} + \ldots.$$

By the definition of a Taylor series, we find that

$$c_{-1} = \operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}[(z-a)^m f(z)]. \tag{4.5.2}$$

Let $z = \infty$ be an isolated singularity of $f(z)$, which is holomorphic in $\mathbb{C} \setminus \overline{D(0, R)}$, for sufficiently large finite R. Then for finite $\rho > R$, the residue at $z = \infty$ is *defined* as (notice the orientation)

$$\operatorname*{Res}_{z=\infty} f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D(0,\rho)} f(z)\, \mathrm{d}z.$$

Let $\zeta = \frac{1}{z}$. Then we get that

$$\begin{aligned}\operatorname*{Res}_{z=\infty} f(z) &= -\frac{1}{2\pi \mathrm{i}} \oint_{\partial D(0,\rho)} f\left(\frac{1}{\zeta}\right) \mathrm{d}\left(\frac{1}{\zeta}\right) \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\partial D(0,1/\rho)} \frac{f(1/\zeta)}{\zeta^2}\, \mathrm{d}\zeta = -\operatorname*{Res}_{\zeta=0} \frac{f(1/\zeta)}{\zeta^2}.\end{aligned}$$

In this definition, if

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \Longleftrightarrow \frac{f(1/\zeta)}{\zeta^2} = \sum_{n=-\infty}^{\infty} c_n \zeta^{-n-2},$$

the residue at $z = \infty$ is equal to $-c_{-1}$. We will later explain the reasoning behind this definition.

Theorem 4.5.1 *(Residue Theorem)*: Let $U \subset \mathbb{C}$ be an open set with a simple closed boundary curve ∂U. Suppose $\{z_n\} \subset U$ is a finite set and $f(z)$ is holomorphic on $U \setminus \{z_n\}$ and continuous on $\overline{U} \setminus \{z_n\}$. Then,

$$\oint_{\partial U} f(z)\, \mathrm{d}z = 2\pi \mathrm{i} \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

Proof: Since U is open, there exists a small disk centered at each isolated singularity z_k of radii δ_k. By the Cauchy–Goursat Theorem (Theorem 3.1.7), we get that

$$\int_{\bigcup_{k=1}^{n} D(z_k,\delta_k)^{-} \cup \partial U^{+}} f(z)\, \mathrm{d}z = 0.$$

From rearrangement, $\oint_{\partial U} f(z)\, \mathrm{d}z = \sum_{k=1}^{n} \oint_{\partial D(z_k,\delta_k)} f(z)\, \mathrm{d}z$, and the conclusion follows. □

This result itself is fairly trivial. Now we will explain the significance of the residue at infinity.

Theorem 4.5.2 *(Global Residue Theorem)*: If $\{z_1, ... z_n, \infty\}$ is discrete and finite, and $f : \hat{\mathbb{C}} \setminus \{z_1, ... z_n, \infty\} \to \mathbb{C}$ is holomorphic, and these points are

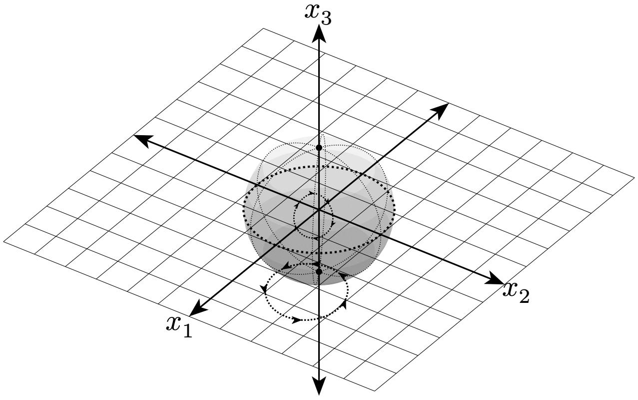

Figure 7: The orientation of a neighborhood that does not enclose ∞ after projection.

the isolated singularities of f, then the sum of the residues at each of these isolated singularities is zero, or

$$\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

Proof: Let $R > \max_{j \in \mathbb{N}_{\leq n}} |z_n|$ be arbitrary. By the Residue Theorem (Theorem 4.5.1),

$$-\operatorname*{Res}_{z=\infty} f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D(0,R)} f(z)\,\mathrm{d}z = \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

as desired. This is merely a restatement of Theorem 4.5.1. □

There is not a directly trivial reason for the definition of the residue at ∞, except for the fact that it seemingly "unifies" the Riemann sphere.

However, if we take a neighborhood of an arbitrary point in $\mathbb{C}$ on the Riemann sphere and traverse its boundary clockwise (from the perspective of outside the sphere), its projection onto $\mathbb{C}$ will be counterclockwise (Figure 7). However, the boundary of a neighborhood of ∞ in S^2 will have a clockwise projection (hence the difference in orientation). We define its equality with the residue of $-\frac{f(1/\zeta)}{\zeta^2}$ at $\zeta = 0$, rather than $f(1/\zeta)$, because we compose the differential form $f(z)\,\mathrm{d}z$ with the inversion, as opposed to $f(z)$.

For any closed rectifiable curve $\gamma \subset U$ (here we are not bound under the assumption of simpleness), the Residue Theorem can be generalized into:

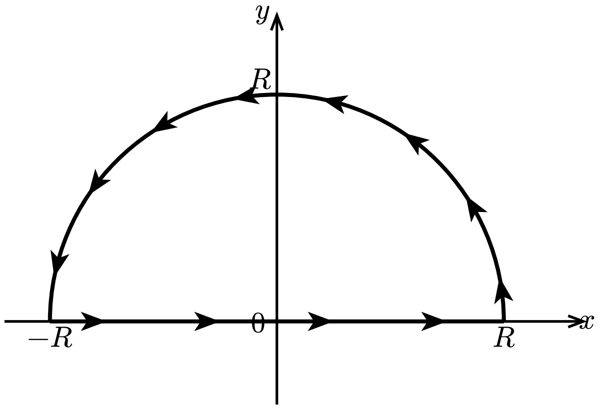

Figure 8: A semicircular contour with orientation marked.

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_k \mathrm{Ind}_\gamma(z_k)\operatorname*{Res}_{z=z_k} f(z)$$

where z_k are the singularities of f in U and Ind_γ is the winding index.

Residues are extremely important as they allow for simple evaluation of definite (most commonly improper) real-valued integrals. This is because oftentimes, residues at poles are generally easy to calculate and have an integral representation. We can integrate over a contour (a smooth closed curve) that contains the important part of the real interval. Oftentimes this is the most non-trivial step.

Example 4.5.1: Evaluate the improper integral $I = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^{n+1}}\,\mathrm{d}x$, where $n \in \mathbb{N}$.

Proof: Consider γ to be a closed semicircle with radius $R \geq 2$ as in Figure 8. Notice that the function $z \mapsto \frac{1}{(z^2+1)^{n+1}}$ has singularities at only $z = \mathrm{i}$ and $z = -\mathrm{i}$, both of which are poles of order $n+1$. By (4.5.2), the residue at $z = \mathrm{i}$ is

$$\begin{aligned}\operatorname*{Res}_{z=\mathrm{i}} \frac{1}{(z^2+1)^{n+1}} &= \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left((z+\mathrm{i})^{-n-1}\right)\bigg|_{z=\mathrm{i}} = \frac{1}{n!}\frac{(-1)^n\prod_{k=1}^n(n+k)}{(2\mathrm{i})^{2n+1}} \\ &= \frac{(-1)^n(2n)!}{(n!)^2(2\mathrm{i})^{2n+1}} = \frac{(2n)!}{2^{2n+1}\mathrm{i}(n!)^2}.\end{aligned}$$

The singularity at $z = -\mathrm{i}$ is not relevant, as it is not enclosed by the contour. By the Residue Theorem (Theorem 4.5.1), we have

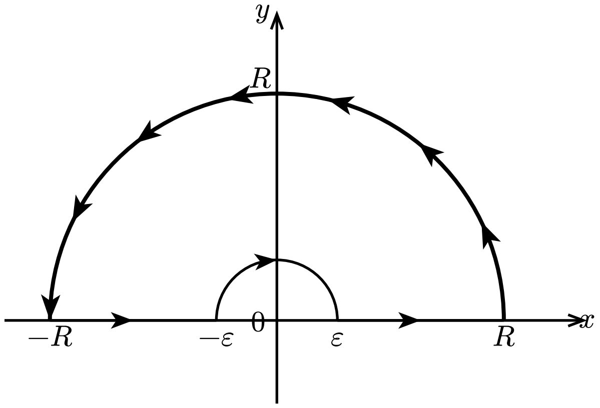

Figure 9: An indented semicircular contour with orientation marked.

$$\begin{aligned}\oint_\gamma \frac{1}{\left(z^2+1\right)^{n+1}}\,\mathrm{d}z &= \int_{-R}^{R} \frac{1}{\left(x^2+1\right)^{n+1}}\,\mathrm{d}x + \int_0^{\pi} \frac{R\mathrm{i}}{\left(R^2\mathrm{e}^{2\mathrm{i}\theta}+1\right)^{n+1}}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \\ &= 2\pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}} \frac{1}{\left(z^2+1\right)^{n+1}} = \frac{(2n)!\pi}{2^{2n}(n!)^2}.\end{aligned}$$

We will now show that the integral over the semicircle vanishes as $R \to \infty$. Under the assumption that $R \geq 2$, since

$$\left|\frac{R\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}}{\left(R^2\mathrm{e}^{2\mathrm{i}\theta}+1\right)^{n+1}}\right| = \frac{R}{\left|R^2\mathrm{e}^{2\mathrm{i}\theta}+1\right|^{n+1}} \leq \frac{R}{|R^2-1|} \leq \frac{2}{3},$$

which is integrable over $[0,\pi]$, and we can commute the limit with the integral. Therefore, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{\left(x^2+1\right)^{n+1}}\,\mathrm{d}x &= \lim_{R\to\infty} \oint_\gamma \frac{1}{\left(z^2+1\right)^{n+1}}\,\mathrm{d}z \\ &\qquad - \int_0^{\pi} \lim_{R\to\infty} \frac{R\mathrm{i}}{\left(R^2\mathrm{e}^{2\mathrm{i}\theta}+1\right)^{n+1}}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \frac{(2n)!\pi}{2^{2n}(n!)^2}.\end{aligned}$$

□

Example 4.5.2 *(Dirichlet Integral)*: Evaluate the integral $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$.

Proof: It is common to use integration with parameters to approach this integral. However, we will now provide a solution via contour integration.

Let $f(z) = \frac{\mathrm{e}^{\mathrm{i}z}}{z}$. Consider a closed contour γ in the form of Figure 9, consisting of a semicircle of radius R in $\overline{\mathbb{H}^+}$ (C_R), a line segment from $-R$ to $-\varepsilon$, a smaller semicircle of radius ε in the upper half-plane (C_ε), and a line segment from ε to R.

By the Cauchy–Goursat Theorem (Theorem 3.1.7), we have that

$$\oint_\gamma f(z)\,\mathrm{d}z = \int_{C_R} f(z)\,\mathrm{d}z + \int_{-R}^{-\varepsilon} f(z)\,\mathrm{d}z + \int_{C_\varepsilon} f(z)\,\mathrm{d}z + \int_{\varepsilon}^{R} f(z)\,\mathrm{d}z = 0.$$

We will now analyze each integral. The first integral is

$$\int_{C_R} f(z)\,\mathrm{d}z = \int_0^\pi \frac{\exp\bigl(\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\bigr)}{R\mathrm{e}^{\mathrm{i}\theta}} R\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \mathrm{i}\int_0^\pi \mathrm{e}^{\mathrm{i}R\cos\theta}\mathrm{e}^{-R\sin\theta}\,\mathrm{d}\theta.$$

Notice that $\frac{2}{\pi}\theta \leq \sin(\theta) \leq \theta$ over the integration range. We want to observe the behavior as $R \to \infty$:

$$\begin{aligned}\left|\mathrm{i}\int_0^\pi \mathrm{e}^{\mathrm{i}R\cos\theta}\mathrm{e}^{-R\sin\theta}\,\mathrm{d}\theta\right| &\leq \int_0^\pi \mathrm{e}^{-R\sin\theta}\,\mathrm{d}\theta = 2\int_0^{\pi/2} \mathrm{e}^{-R\sin\theta}\,\mathrm{d}\theta \\ &< 2\int_0^{\pi/2} \mathrm{e}^{-R\frac{2}{\pi}\theta}\,\mathrm{d}\theta = -\frac{\pi}{R}\mathrm{e}^{-R\frac{2}{\pi}\theta}\Big|_0^{\pi/2} \\ &= \frac{\pi}{R}\bigl(1-\mathrm{e}^{-R}\bigr) \to 0.\end{aligned}$$

Let us evaluate the integral on γ_ε as $\varepsilon \to 0$:

$$\int_{C_\varepsilon} f(z)\,\mathrm{d}z = \mathrm{i}\int_\pi^0 \exp(\varepsilon(\mathrm{i}\cos\theta - \sin\theta))\,\mathrm{d}\theta = \mathrm{i}\int_\pi^0 \mathrm{e}^{-\varepsilon\sin\theta}\mathrm{e}^{\mathrm{i}\varepsilon\cos\theta}\,\mathrm{d}\theta.$$

Obviously,

$$\left|\mathrm{e}^{-\varepsilon\sin\theta}\mathrm{e}^{\mathrm{i}\varepsilon\cos\theta}\right| \leq 1,$$

and therefore, the integral and the limit may commute:

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} f(z)\,\mathrm{d}z = \mathrm{i}\int_\pi^0 \lim_{\varepsilon\to 0^+} \mathrm{e}^{-\varepsilon\sin\theta}\mathrm{e}^{\mathrm{i}\varepsilon\cos\theta}\,\mathrm{d}\theta = \mathrm{i}\int_\pi^0 \mathrm{d}\theta = -\mathrm{i}\pi.$$

Evaluating the integral over the line segments, we have

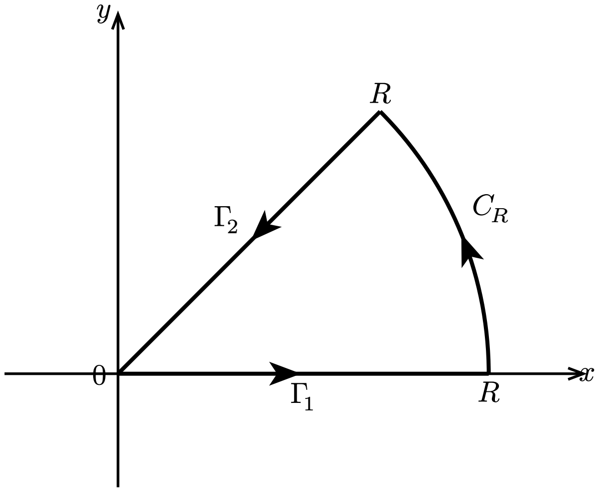

Figure 10: A wedge contour with orientation marked.

$$\begin{aligned}\int_{-R}^{-\varepsilon} f(z)\,\mathrm{d}z + \int_{\varepsilon}^{R} f(z)\,\mathrm{d}z &= \int_{-R}^{-\varepsilon} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z \\ &= \int_{-R}^{-\varepsilon} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z - \int_{-R}^{-\varepsilon} \frac{\mathrm{e}^{-\mathrm{i}z}}{z}\,\mathrm{d}z \\ &\to 2\mathrm{i}\int_{-\infty}^{0} \frac{\sin z}{z}\,\mathrm{d}z = 2\mathrm{i}\int_{0}^{\infty} \frac{\sin z}{z}\,\mathrm{d}z.\end{aligned}$$

Hence,

$$-\mathrm{i}\pi + 2\mathrm{i}\int_0^\infty \frac{\sin(z)}{z}\,\mathrm{d}z = 0 \Longleftrightarrow \int_0^\infty \frac{\sin z}{z}\,\mathrm{d}z = \frac{\pi}{2}. \qquad \square$$

Example 4.5.3 *(Fresnel Integral)*: Evaluate the improper integrals

$$I_1 = \int_0^\infty \cos(x^2)\,\mathrm{d}x, \quad I_2 = \int_0^\infty \sin(x^2)\,\mathrm{d}x.$$

Proof: Let $f(z) = \mathrm{e}^{\mathrm{i}z^2}$. Choose the wedge contour composed of

$$\Gamma_1 = \{x \in \mathbb{R} : 0 \le x \le R\}, \quad \Gamma_2 = \left\{r\mathrm{e}^{\mathrm{i}\pi/4} : 0 \le r \le R\right\},$$
$$C_R = \left\{R\mathrm{e}^{\mathrm{i}\theta} : 0 \le \theta \le \frac{\pi}{4}\right\}$$

as in Figure 10. By the Cauchy–Goursat Theorem (Theorem 3.1.7), we have that

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z + \int_{\Gamma_2} f(z)\,\mathrm{d}z + \int_{C_R} f(z)\,\mathrm{d}z = 0. \tag{4.5.3}$$

The third integral can be written as

$$\int_{C_R} f(z)\,\mathrm{d}z = R\mathrm{i} \int_0^{\pi/4} \exp\left[\mathrm{i}\left(R\mathrm{e}^{\mathrm{i}\theta}\right)^2\right] \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta.$$

Using the fact that $\frac{4}{\pi}\theta < \sin(2\theta)$ on the integration range, it can be bounded as

$$\begin{aligned}\left|\int_{C_R} f(z)\,\mathrm{d}z\right| &\leq R\int_0^{\pi/4} \mathrm{e}^{-R^2\sin(2\theta)}\,\mathrm{d}\theta < R\int_0^{\pi/4} \mathrm{e}^{-\frac{4}{\pi}R^2\theta}\,\mathrm{d}\theta \\ &= -\frac{\pi}{4R}\mathrm{e}^{-\frac{4}{\pi}R^2\theta}\Big|_0^{\pi/4} = \frac{\pi}{4R}\left(1-\mathrm{e}^{-R^2}\right).\end{aligned}$$

As $R \to \infty$, this integral tends to 0. Let $z = r\mathrm{e}^{\mathrm{i}\pi/4}$ on Γ_2. Then, we have

$$\lim_{R\to\infty}\int_{\Gamma_2} f(z)\,\mathrm{d}z = \int_\infty^0 \exp\left[\mathrm{i}\left(r\mathrm{e}^{\mathrm{i}\pi/4}\right)^2\right]\mathrm{e}^{\mathrm{i}\pi/4}\,\mathrm{d}r = \mathrm{e}^{\mathrm{i}\pi/4}\int_\infty^0 \exp(-r^2)\,\mathrm{d}r.$$

From (4.5.3), we have that

$$\int_0^\infty \mathrm{e}^{\mathrm{i}r^2}\,\mathrm{d}r = \mathrm{e}^{\mathrm{i}\pi/4}\int_0^\infty \mathrm{e}^{-r^2}\,\mathrm{d}r.$$

Since $\int_0^\infty \mathrm{e}^{-r^2}\,\mathrm{d}r = \frac{\sqrt{\pi}}{2}$, we have

$$\int_0^\infty \mathrm{e}^{\mathrm{i}r^2}\,\mathrm{d}r = \left(\frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}\right)\frac{\sqrt{\pi}}{2}.$$

Since $\mathrm{e}^{\mathrm{i}r^2} = \cos(r^2) + \mathrm{i}\sin(r^2)$, we have

$$\begin{aligned}\int_0^\infty \cos(r^2)\,\mathrm{d}r &= \mathfrak{Re}\left[\int_0^\infty \mathrm{e}^{\mathrm{i}r^2}\,\mathrm{d}r\right] = \frac{\sqrt{2\pi}}{4}, \\ \int_0^\infty \sin(r^2)\,\mathrm{d}r &= \mathfrak{Im}\left[\int_0^\infty \mathrm{e}^{\mathrm{i}r^2}\,\mathrm{d}r\right] = \frac{\sqrt{2\pi}}{4},\end{aligned}$$

as desired. ☐

Example 4.5.4: Evaluate the integrals $\int_0^{2\pi} \Phi(\cos\theta, \sin\theta)\,\mathrm{d}\theta$, where $\Phi(\xi, \eta)$ is a rational function of ξ and η that is continuous on $\theta \in [0, 2\pi]$.

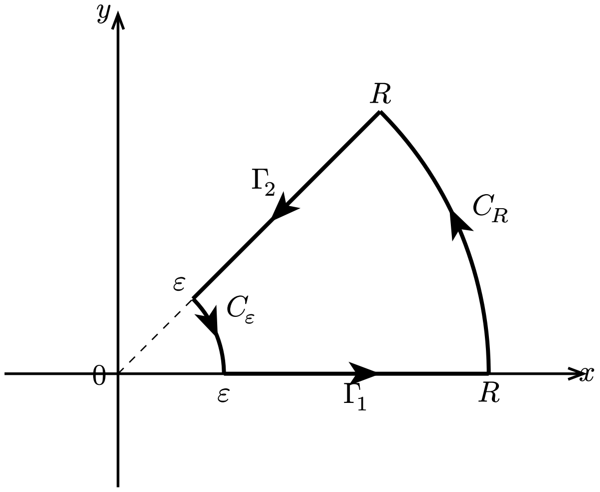

Figure 11: An indented wedge contour with orientation marked.

Proof: Let $z = \mathrm{e}^{\mathrm{i}\theta}$. Consequently, we have $\cos\theta = \frac{z+z^{-1}}{2}$, $\sin\theta = \frac{z-z^{-1}}{2\mathrm{i}}$, and $\mathrm{d}z = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta$, implying that $\mathrm{d}\theta = \frac{\mathrm{d}z}{\mathrm{i}z}$. Therefore, by the Residue Theorem (Theorem 4.5.1), letting $f(z) = \frac{1}{\mathrm{i}z}\Phi\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2\mathrm{i}}\right)$, we have

$$\int_0^{2\pi} \Phi(\cos\theta, \sin\theta)\,\mathrm{d}\theta = \oint_{\partial\mathbb{D}} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z),$$

where z_k where $k = 1, ..., n$ are the isolated singularities of f in $\mathbb{D}$. □

Example 4.5.5: Evaluate $I = \int_0^\infty \frac{x^\alpha}{1+x^\beta}\,\mathrm{d}x$, where $0 < \alpha + 1 < \beta$.

Proof: Let $f(z) = \frac{z^\alpha}{1+z^\beta}$ and let $-\pi < \operatorname{Arg}(z) \leq \pi$ in the principal branches of $z^\alpha = \mathrm{e}^{\alpha\operatorname{Log}(z)}$ and $z^\beta = \mathrm{e}^{\beta\operatorname{Log}(z)}$. Then except for at the zeros of $1 + z^\beta$, f is holomorphic.

The solutions to $z^\beta = -1$ are $z = \exp\left(\mathrm{i}\frac{\pi}{\beta} + 2\mathrm{i}k\frac{\pi}{\beta}\right)$. Choose an indented wedge contour (as there is a logarithmic branch point singularity at the origin) with an angle of $2\frac{\pi}{\beta}$ (as in Figure 10). The only singularity it encloses is $\exp\left(\mathrm{i}\frac{\pi}{\beta}\right)$. Since it is a simple zero of $\frac{1}{f}$, this singularity is a simple pole.

The contour is the union of the following curves:

$$\Gamma_1 = \{x \in \mathbb{R} : \varepsilon \leq x \leq R\}, \quad \Gamma_2 = \{r\exp(\mathrm{i}2\pi/\beta) : \varepsilon \leq r \leq R\},$$
$$C_R = \left\{R\mathrm{e}^{\mathrm{i}\theta} : 0 \leq \theta \leq 2\pi/\beta\right\}, \quad C_\varepsilon = \left\{\varepsilon\mathrm{e}^{\mathrm{i}\theta} : 0 \leq \theta \leq 2\pi/\beta\right\}$$

where $R > 1$ and $0 < \varepsilon < 1$. By the Residue Theorem (Theorem 4.5.1), we get that

$$\lim_{\varepsilon\to 0}\lim_{R\to\infty}\left(\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{C_R} + \int_{C_\varepsilon}\right) f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}\left[f, \exp\left(\mathrm{i}\frac{\pi}{\beta}\right)\right].$$

By (4.5.2), it follows that

$$\begin{aligned}\mathrm{Res}[f, \exp(\mathrm{i}\pi/\beta)] &= \lim_{z\to\exp(\mathrm{i}\pi/\beta)} \frac{z - \exp(\mathrm{i}\pi/\beta)}{(1+z^\beta)/z^\alpha} \\ &= \lim_{z\to\exp(\mathrm{i}\pi/\beta)} \left(\frac{\mathrm{d}}{\mathrm{d}z}\big(z^{-\alpha} + z^{\beta-\alpha}\big)\right)^{-1} \\ &= \lim_{z\to\exp(\mathrm{i}\pi/\beta)} \frac{z^{\alpha+1}}{(\beta-\alpha)z^\beta - \alpha} \\ &= -\frac{1}{\beta}\exp\left(\mathrm{i}\frac{\pi}{\beta}(\alpha+1)\right).\end{aligned}$$

We can write the integral on Γ_2 in terms of I:

$$\begin{aligned}\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{\Gamma_2} f(z)\,\mathrm{d}z &= \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_R^0 f[r\exp(\mathrm{i}2\pi/\beta)]\exp(\mathrm{i}2\pi/\beta)\,\mathrm{d}r \\ &= -\exp\left[\mathrm{i}2\frac{\pi}{\beta}(1+\alpha)\right]\int_0^\infty \frac{r^\alpha}{1+r^\beta}\,\mathrm{d}r.\end{aligned}$$

We also have

$$\int_{C_R} f(z)\,\mathrm{d}z = R\mathrm{i}\int_0^{2\pi/\beta} f\big(R\mathrm{e}^{\mathrm{i}\theta}\big)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \mathrm{i}\int_0^{2\pi/\beta} \frac{R^{\alpha+1}}{1+R^\beta\mathrm{e}^{\mathrm{i}\beta\theta}}\exp[\mathrm{i}\theta(1+\alpha)]\,\mathrm{d}\theta.$$

It can also be shown that the integral is bounded by a vanishing function as $R\to\infty$:

$$\left|\int_0^{2\pi/\beta} \frac{R^{\alpha+1}}{1+R^\beta\mathrm{e}^{\mathrm{i}\beta\theta}}\exp[\mathrm{i}\theta(1+\alpha)]\,\mathrm{d}\theta\right| \le \int_0^{2\pi/\beta}\frac{R^{\alpha+1}}{R^\beta - 1}\,\mathrm{d}\theta = 2\frac{\pi}{\beta}\frac{R^{\alpha+1}}{R^\beta-1} \to 0.$$

Similarly, as $\varepsilon\to 0$,

$$\left|\int_{C_\varepsilon} f(z)\,\mathrm{d}z\right| \le \varepsilon\int_0^{2\pi/\beta}\big|f\big(\varepsilon\mathrm{e}^{\mathrm{i}\theta}\big)\big|\,\mathrm{d}\theta = \int_0^{2\pi/\beta}\frac{\varepsilon^{\alpha+1}}{1-\varepsilon^\beta}\,\mathrm{d}\theta = 2\frac{\pi}{\beta}\frac{\varepsilon^{\alpha+1}}{1-\varepsilon^\beta}\to 0.$$

By letting $R\to\infty$ and $\varepsilon\to 0$, we have

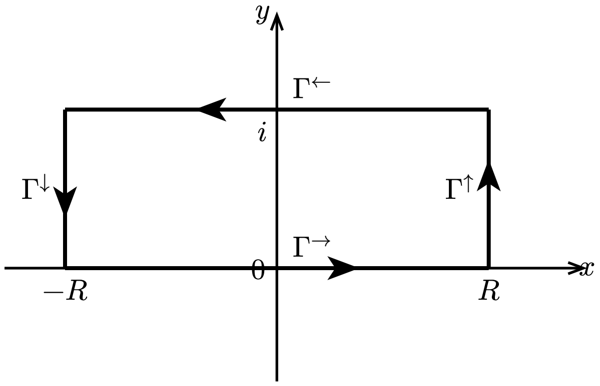

Figure 12: A rectangular contour with orientation marked.

$$\left[1-\exp\left(\mathrm{i}2\frac{\pi}{\beta}(1+\alpha)\right)\right]I=-\frac{2\pi\mathrm{i}}{\beta}\exp\left(\mathrm{i}\frac{\pi}{\beta}(\alpha+1)\right).$$

It follows that

$$I=\frac{2\pi\mathrm{i}}{\beta}\left[\exp\left(\mathrm{i}\frac{\pi}{\beta}(\alpha+1)\right)-\exp\left(-\mathrm{i}\frac{\pi}{\beta}(\alpha+1)\right)\right]^{-1}=\frac{\pi}{\beta}\csc\left(\frac{\pi}{\beta}(\alpha+1)\right)\square$$

Example 4.5.6: Prove that the Fourier transform of $\operatorname{sech}(\pi x)$ is itself, or that

$$I(\xi)=\int_{-\infty}^{\infty}\exp(-2\pi\mathrm{i}x\xi)\operatorname{sech}(\pi x)\,\mathrm{d}x=\operatorname{sech}(\pi\xi).$$

Proof: Fix $\xi\in\mathbb{R}$ and let $f(z)=\frac{\exp(-2\pi\mathrm{i}z\xi)}{\cosh(\pi z)}$. Its poles in $\mathbb{C}$ occur when $\mathrm{e}^{\pi z}+\mathrm{e}^{-\pi z}=0$, or equivalently, when $z=\mathrm{i}\left(n+\frac{1}{2}\right)$, where $n\in\mathbb{Z}$.

Since

$$\cosh(\pi(z+\mathrm{i}))=-\cosh(\pi z),\quad \exp(-2\pi\mathrm{i}(z+\mathrm{i})\xi)=\exp(2\pi\xi)\exp(-2\pi\mathrm{i}z\xi),$$

we have that $f(z)$ is a constant multiple of $f(z+\mathrm{i})$. In particular, $f(z+\mathrm{i})=-\exp(2\pi\xi)f(z)$. Therefore, we can use a rectangular contour as shown in Figure 12. Let the sides be denoted by

$$\Gamma^{\leftarrow}=\{x+\mathrm{i}:-R\le x\le R,x\in\mathbb{R}\},\quad \Gamma^{\rightarrow}=\{x\in\mathbb{R}:-R\le x\le R\}$$
$$\Gamma^{\downarrow}=\{-R+\mathrm{i}y:y\in[0,1]\},\quad \Gamma^{\uparrow}=\{R+\mathrm{i}y:y\in[0,1]\}.$$

The only enclosed singularity is a simple pole at $z=\frac{\mathrm{i}}{2}$ (simple by evaluation of the Taylor expansion of the denominator). By the Residue Theorem (Theorem 4.5.1), we get that

$$\left(\int_{\Gamma^{\rightarrow}} + \int_{\Gamma^{\uparrow}} + \int_{\Gamma^{\leftarrow}} + \int_{\Gamma^{\downarrow}}\right) f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\operatorname{Res}\left(f, \frac{\mathrm{i}}{2}\right). \tag{4.5.4}$$

By (4.5.2), we have

$$\begin{aligned} \operatorname{Res}\left(f, \frac{\mathrm{i}}{2}\right) &= \lim_{z\to\mathrm{i}/2}\left(z - \frac{\mathrm{i}}{2}\right)\frac{\exp(-2\pi\mathrm{i}z\xi)}{\cosh(\pi z)} \\ &= \lim_{z\to\mathrm{i}/2} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\cosh(\pi z)}{\exp(-2\pi\mathrm{i}z\xi)}\right)^{-1} \\ &= \lim_{z\to\mathrm{i}/2} \frac{\exp(-2\pi\mathrm{i}z\xi)}{\pi\sinh(\pi z) + 2\pi\mathrm{i}\xi\cosh(\pi z)} \\ &= \frac{\exp(\pi\xi)}{\pi\mathrm{i}}. \end{aligned}$$

The sum of the horizontal line integrals is equal to

$$\begin{aligned} \int_{-R}^{R} f(z)\,\mathrm{d}z + \int_{R}^{-R} f(z+\mathrm{i})\,\mathrm{d}z &= \int_{-R}^{R} f(z)\,\mathrm{d}z - \int_{R}^{-R} \mathrm{e}^{2\pi\xi} f(z)\,\mathrm{d}z \\ &= \left(1 + \mathrm{e}^{2\pi\xi}\right)\int_{-R}^{R} f(z)\,\mathrm{d}z. \end{aligned}$$

As $R \to \infty$, we have $\int_{\Gamma^{\rightarrow}} f(z)\,\mathrm{d}z + \int_{\Gamma^{\leftarrow}} f(z)\,\mathrm{d}z \to \left(1 + \mathrm{e}^{2\pi\xi}\right)I(\xi)$. The remaining two integrals can be written as

$$\begin{aligned} \int_{\Gamma^{\uparrow}} f(z)\,\mathrm{d}z &= \int_{0}^{1} \frac{\exp(-2\pi\mathrm{i}(R+\mathrm{i}z)\xi)}{\cosh(\pi(R+\mathrm{i}z))}\,\mathrm{d}z \\ \int_{\Gamma^{\downarrow}} f(z)\,\mathrm{d}z &= \int_{1}^{0} \frac{\exp(2\pi\mathrm{i}(R-\mathrm{i}z)\xi)}{\cosh(\pi(-R+\mathrm{i}z))}\,\mathrm{d}z. \end{aligned}$$

They can be bounded with

$$\begin{aligned} \left|\int_{0}^{1} \frac{\exp(-2\pi\mathrm{i}(R+\mathrm{i}z)\xi)}{\cosh(\pi(R+\mathrm{i}z))}\,\mathrm{d}z\right| &\le 2\int_{0}^{1} \frac{\exp(2\pi z\xi)}{|\mathrm{e}^{\pi R}\mathrm{e}^{\pi\mathrm{i}z} + \mathrm{e}^{-\pi R}\mathrm{e}^{-\pi\mathrm{i}z}|}\,\mathrm{d}z \\ &\le 2\int_{0}^{1} \frac{\exp(2\pi z\xi)}{|\mathrm{e}^{\pi R} - \mathrm{e}^{-\pi R}|}\,\mathrm{d}z \end{aligned}$$

and

$$\left|\int_1^0 \frac{\exp(2\pi\mathrm{i}(R-\mathrm{i}z)\xi)}{\cosh(\pi(-R+\mathrm{i}z))}\,\mathrm{d}z\right| \le 2\int_0^1 \frac{\exp(2\pi z\xi)}{|\mathrm{e}^{-\pi R}\mathrm{e}^{\pi\mathrm{i}z}+\mathrm{e}^{\pi R}\mathrm{e}^{-\pi\mathrm{i}z}|}\,\mathrm{d}z$$
$$\le 2\int_0^1 \frac{\exp(2\pi z\xi)}{|\mathrm{e}^{\pi R}-\mathrm{e}^{-\pi R}|}\,\mathrm{d}z.$$

Since the integrands are continuous and uniformly convergent to 0 with respect to z, we have

$$\int_{\Gamma^{\uparrow}} f(z)\,\mathrm{d}z + \int_{\Gamma^{\downarrow}} f(z)\,\mathrm{d}z \to 0$$

as $R \to \infty$. By rearrangement of (4.5.4),

$$I(\xi)\big(1+\mathrm{e}^{2\pi\xi}\big) = 2\exp(\pi\xi),$$

or that

$$I(\xi) = \frac{2}{\mathrm{e}^{-\pi\xi}+\mathrm{e}^{\pi\xi}} = \operatorname{sech}(\pi\xi),$$

which proves the result. □

Contour integration provides a powerful method for evaluating real improper integrals by leveraging the Residue Theorem (Theorem 4.5.1). The primary challenge often lies in constructing a suitable contour in the complex plane that encloses the relevant singularities of the integrand f while ensuring that the contribution from the contributions from the remaining segments of the contour either vanishes or can be calculated with ease.

If the function f is even and integrated on a domain such as $\mathbb{R}_{\ge 0}$, then the integral can be extended to the entire real axis. If f decays sufficiently rapidly in the upper half plane $\mathbb{H}^+$, a semicircular contour is generally preferable, as illustrated in Figure 8. In the presence of singularities on the contour itself, we can insert arc indentations around them, as shown in Figure 9.

If $f(z)$ is a constant multiple of $f(z+\mathrm{i}y)$ (a type of quasiperiodicity) for some $y \in \mathbb{R}$, it is a strong indication to use a rectangular contour. If $f(z)$ is a constant multiple of $f(z\mathrm{e}^{\mathrm{i}\tau})$ for some $\tau \in \mathbb{R}$, a wedge-shaped contour is an appropriate choice.

In the case that there are indentations along the contour, we have

Theorem 4.5.3: Let $\lambda > 0$ and let $a \in \mathbb{C}$. Suppose $f(z)$ is a holomorphic function on $D^*(a,\lambda)$ with a simple pole at $z = a \in U$. Let $0 < \varepsilon < \lambda$ and define $\gamma_\varepsilon \subseteq \partial D(a,\varepsilon)$ be a counterclockwise-oriented, connected arc subtending an angle ϑ. Then,

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = \mathrm{i}\vartheta \cdot \operatorname*{Res}_{z=a} f(z).$$

Proof: Parameterize γ_ε with $z = a + \varepsilon \mathrm{e}^{\mathrm{i}\theta}$, where $\theta \in [\alpha,\beta]$ and $\beta - \alpha = \vartheta$. Then,

$$\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = \int_\alpha^\beta f\big(a+\varepsilon\mathrm{e}^{\mathrm{i}\theta}\big)\frac{\mathrm{d}z}{\mathrm{d}\theta}\,\mathrm{d}\theta = \varepsilon\mathrm{i}\int_\alpha^\beta f\big(a+\varepsilon\mathrm{e}^{\mathrm{i}\theta}\big)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta.$$

Since f has a simple pole at $z = a$, we can write a Laurent expansion around a as

$$f(z) = \frac{c_{-1}}{z-a} + \varphi(z),$$

where $\varphi(z)$ is holomorphic in a neighborhood of a and $c_{-1} = \operatorname{Res}_{z=a} f(z)$. Then for $z = a + \varepsilon\mathrm{e}^{\mathrm{i}\theta}$,

$$f\big(a+\varepsilon\mathrm{e}^{\mathrm{i}\theta}\big) = \frac{c_{-1}}{\varepsilon\mathrm{e}^{\mathrm{i}\theta}} + \varphi\big(a+\varepsilon\mathrm{e}^{\mathrm{i}\theta}\big).$$

So,

$$\begin{aligned}\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z &= \varepsilon\mathrm{i}\int_\alpha^\beta \left(\frac{c_{-1}}{\varepsilon\mathrm{e}^{\mathrm{i}\theta}} + \varphi\big(a+\varepsilon\mathrm{e}^{\mathrm{i}\theta}\big)\right)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \\ &= \mathrm{i}c_{-1}\int_\alpha^\beta \mathrm{d}\theta + \varepsilon\mathrm{i}\int_\alpha^\beta \varphi\big(a+\varepsilon\mathrm{e}^{\mathrm{i}\theta}\big)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \\ &= \mathrm{i}c_{-1}\vartheta + \varepsilon\mathrm{i}\int_\alpha^\beta \varphi\big(a+\varepsilon\mathrm{e}^{\mathrm{i}\theta}\big)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta.\end{aligned}$$

Let $\varepsilon < \frac{\lambda}{2}$. Since φ is continuous on the disk $\overline{D\big(a,\frac{\lambda}{2}\big)}$, it is bounded. Therefore, letting $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0}\varepsilon\mathrm{i}\int_\alpha^\beta \varphi\big(a+\varepsilon\mathrm{e}^{\mathrm{i}\theta}\big)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \lim_{\varepsilon\to 0}\varepsilon\mathrm{i}\int_\alpha^\beta \varphi(a)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = 0.$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = \mathrm{i}\vartheta\operatorname*{Res}_{z=a} f(z). \qquad \square$$

In the case that a branch point singularity is present on the contour, we may attempt to rewrite the function in a way such that the branch point is irrelevant. Otherwise, there are two types of “keyhole contours” that can be used to avoid the branch cut.

Example 4.5.7: Evaluate $I = \int_0^\infty \frac{\log(x^2+1)}{x^2+1}\,\mathrm{d}x$.

Proof: Notice that the integrand itself has branch points at $z = \pm\mathrm{i}$ coinciding with the poles from the denominator. We can rewrite the integral as

$$\begin{aligned} I &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2+1} = \int_{-\infty}^{\infty} \frac{\log\sqrt{(x+i)(x-i)}}{x^2+1} \\ &= \int_{-\infty}^{\infty} \frac{\log|x \pm i|}{x^2+1} = \mathfrak{Re}\int_{-\infty}^{\infty} \frac{\log(x+i)}{x^2+1}. \end{aligned} \tag{4.5.5}$$

Let $\gamma = \Gamma \cup C_R$, where concretely,

$$\Gamma = \{x \in \mathbb{R} : -R \le x \le R\}, \quad C_R = \left\{R\mathrm{e}^{\mathrm{i}\theta} : 0 \le \theta \le \pi\right\}$$

and $R > 2$, and let $f(z) = \frac{\operatorname{Log}(z+\mathrm{i})}{z^2+1}$, where the branch for Log is chosen to satisfy $[0, \pi] \subset \mathfrak{Im}\log(\mathbb{C}^*)$, such as the principal branch. The only singularity of f in the upper half plane is a simple pole at $z = \mathrm{i}$. By the Residue Theorem (Theorem 4.5.1), we have

$$\lim_{R\to\infty}\oint_\gamma f(z)\,\mathrm{d}z = \lim_{R\to\infty}\left(\int_\Gamma + \int_{C_R}\right) f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}} f(z).$$

By (4.5.2), we have

$$\operatorname*{Res}_{z=\mathrm{i}} f(z) = \lim_{z\to\mathrm{i}}(z-\mathrm{i})\frac{\log(z+\mathrm{i})}{z^2+1} = \lim_{z\to\mathrm{i}}\frac{\log(z+\mathrm{i})}{z+\mathrm{i}} = \frac{\log(2\mathrm{i})}{2\mathrm{i}} = \frac{\pi}{4} - \mathrm{i}\frac{\log(2)}{2}.$$

Additionally, for $z \in C_R$, since as $R \to \infty$, $|f(z)| = \left|\frac{\operatorname{Log}(z+i)}{z^2+1}\right| \le \frac{|\log|z+i||+\pi}{R^2-1} \le \frac{\log|R+1|+\pi}{R^2-1} < \frac{R+1+\pi}{R^2-1} \to 0$ by virtue of $R > 2$, it follows that $\int_{C_R} f(z)\,\mathrm{d}z \to 0$.

Since $\lim_{R\to\infty}\int_\Gamma f(z)\,\mathrm{d}z = \int_{-\infty}^{\infty} f(z)\,\mathrm{d}z$ and

$$\int_{-\infty}^{\infty} f(z)\,\mathrm{d}z = \frac{\pi^2\mathrm{i}}{2} + \pi\log(2),$$

by (4.5.5), we have $I = \mathfrak{Re}\int_{-\infty}^{\infty} f(z)\,\mathrm{d}z = \pi\log(2)$. □

5 The Geometric Theory of Conformal Mappings

5.1 Biholomorphy

In Section 2.4, it was asserted that for a holomorphic function $f(z)$, the map $w = f(z)$ is conformal when $f'(z) \neq 0$.

We have the following immediate assertion:

Theorem 5.1.1 *(Open Mapping Theorem)*: Suppose $U \subseteq \mathbb{C}$ is a region (open, nonempty, and connected). Then the image of any holomorphic and non-constant function $f : U \to \mathbb{C}$, $f(U)$, is a region.

Proof: The nonemptiness of $f(U)$ is an immediate conclusion from the fact that U is nonempty and f is defined on all of U.

Let w_0 be an arbitrary point in $f(U)$. Then $\exists z_0 \in U$ such that $f(z_0) = w_0$. Since f is non-constant, the function $f - w_0$ has an isolated zero at z_0. Thus for sufficiently small $\rho > 0$, the only zero of $f - w_0$ in $\overline{D(z_0, \rho)}$ is at z_0.

By Theorem 3.3.7, then there exists $\delta > 0$ such that $\forall \varepsilon \in D(0, \delta)$, $f(z) - w_0 - \varepsilon$ has exactly one zero in $\overline{D(z_0, \rho)}$. In other words, $\forall w_0 \in f(U)$, $\exists \delta > 0$ such that $\forall w \in D(w_0, \delta)$, $\exists! z \in \overline{D(z_0, \rho)}$ such that $f(z) = w$. Thus, $D(w_0, \delta) \subseteq f(U)$. Thus, $f(U)$ is an open set since each contained point has a fully contained open neighborhood.

Let $w_1, w_2 \in f(U)$ be arbitrary and distinct. Then there exist $z_1, z_2 \in U$ such that $f(z_1) = w_1$ and $f(z_2) = w_2$. By the connectivity of U, there exists a path $\gamma \subset U$ that connects z_1 and z_2. Then $f(\gamma) \subset f(U)$ is a curve that joins w_1 and w_2. Thus, $f(U)$ is connected. □

Holomorphic injectivity, or univalence, satisfies the proceeding assertion:

Lemma 5.1.1: Let $U \subseteq \mathbb{C}$ be a region and suppose $f : U \to \mathbb{C}$ is univalent. Then f' is non-vanishing on U.

Proof: Suppose, for the sake of contradiction, that f is univalent on U such that $\exists z_0 \in U$ such that $f'(z_0) = 0$. Let $w_0 = f(z_0)$. The previous statement is equivalent to: $f(z) - w_0$ has a zero at z_0 with multiplicity $m \geq 2$.

Since f is univalent, neither $f - w_0$ nor f' may have accumulation points in U. Thus, $\exists \rho > 0$ such that z_0 is the only zero of either $f - w_0$ and f' contained in $\overline{D(z_0, \rho)} \subset U$. By Theorem 3.3.7, $\exists \delta > 0$ such that $\forall w \in \overline{D^*(w_0, \delta)} = D(w_0, \delta) \setminus \{w_0\}$, the equation $f(z) = w$ has m solutions in $\overline{D(z_0, \rho)}$, which cannot lie all at a single point (unless that point is z_0 itself, which cannot be the case as z_0 already maps to $w_0 \neq w$), as otherwise z_0

would not be the only zero of f' in $\overline{D(z_0, \rho)}$. This contradicts the univalence of f. $\square$

Conversely, we have the following statement on local univalence and invertibility.

Theorem 5.1.2: Let $U \subseteq \mathbb{C}$ be a region and suppose $f : U \to \mathbb{C}$ is holomorphic. If $f'(z_0) \neq 0$ for some $z_0 \in U$, then there exists an open neighborhood of z_0 on which f is univalent.

Proof: Let $w_0 = f(z_0)$. Since $\lim_{z \to z_0} f(z) - w_0 = 0$ and $\lim_{z \to z_0} \frac{f(z)-w_0}{z-z_0} \neq 0$, it follows that z_0 is a simple zero of $f(z) - w_0$. Let V be an open neighborhood (relatively compact in U) of z_0 whose closure does not contain other zeros of $f - w_0$. By Theorem 3.3.7, $\exists \delta > 0$ such that $\forall w \in D(w_0, \delta)$, $f(z) = w$ has only one solution for z satisfying $z \in V$. Therefore, we can choose a relatively compact open subset W of V such that $f(W) \subseteq D(w_0, \delta)$, on which f is univalent. $\square$

Moreover, if $w = f(z)$ is univalent and surjective, mapping U to G, then its inverse $z = f^{-1}(w)$ is univalent on G. Such bijective holomorphic functions are known as *biholomorphisms* or *biholomorphic* functions.

We will now study holomorphic functions from a more geometric perspective.

Theorem 5.1.3: Let $\Omega \subseteq \mathbb{C}$ be a region, and let $\gamma \subset \Omega$ be a rectifiable simple closed counterclockwise-oriented curve that is null-homotopic in Ω. Denote $\operatorname{int}(\gamma)$ by U. If $f : \Omega \to \mathbb{C}$ is holomorphic and maps γ injectively to a simple closed curve Γ, then $w = f(z)$ is univalent in U, $f(U) = \operatorname{int}(\Gamma)$, and Γ is traversed counterclockwise.

Proof: Let $w_0 \in \mathbb{C}$. Let $k = k(w_0)$ be the number of zeros of $f - w_0$ in U. By the Argument Principle (Theorem 3.3.4), for $w_0 \notin \Gamma$,

$$k = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z) - w_0} \, dz = \frac{1}{2\pi i} \oint_\Gamma \frac{dw}{w - w_0} = \operatorname{Ind}_\Gamma(w_0).$$

1 If $w_0 \in \operatorname{ext}(\Gamma)$, the expression vanishes since $\operatorname{Ind}_\Gamma(w_0) = 0$. Then $f(z) = w_0$ has no solution in U (i.e. $k = 0$).

2 If $w_0 \in \operatorname{int}(\Gamma)$, then Γ winds around w_0 exactly once, and hence, in other words, $\forall w_0 \in \operatorname{int}(\Gamma)$, $f(z) = w_0$ has a unique solution in U (i.e. $k = 1$).

3 If w_0 lies on Γ, then it can be shown that $f - w_0$ has no zeros in U.

Indeed, for the sake of contradiction, assume that $\exists z_0 \in U$ such that $f(z_0) = w_0$. By the Open Mapping Theorem (Theorem 5.1.1), $\exists \delta > 0$ such that $D(w_0, \delta) \subseteq f(U)$, or equivalently, $\forall w \in D(w_0, \delta)$, $f - w$ has zeros in

U. Since w_0 lies on Γ, a subset of $D(w_0, \delta)$ lies in the exterior of Γ. It was previously established that $f - w$ has no zeros if $w \in D(w_0, \delta) \cap \operatorname{ext}(\Gamma)$. Thus, we have a contradiction, and no such z_0 exists, implying $k = 0$.

We then have

$$k = \begin{cases} 0 \text{ if } w_0 \in \overline{\operatorname{ext}(\Gamma)} \\ 1 \text{ if } w_0 \in \operatorname{int}(\Gamma) \end{cases}.$$

Hence, f is univalent in U (since for each w_0, k is at most one).

Moreover, any point $z_0 \in U$ must map to either $\operatorname{int}(\Gamma)$ or $\overline{\operatorname{ext}(\Gamma)}$. The latter is an impossibility since otherwise $k \neq 0$. This $f(U) = \operatorname{int}(\Gamma)$. □

We will now give examples of biholomorphisms.

Example 5.1.1: The only biholomorphisms which map $\mathbb{D}$ to itself are in the form of

$$w = \mathrm{e}^{\mathrm{i}\theta} \frac{z - a}{1 - \overline{a} z}, \quad a \in \mathbb{D}, \theta \in \mathbb{R}. \tag{5.1.1}$$

This follows directly from Theorem 3.5.1.

Example 5.1.2: The only biholomorphisms which map $\mathbb{H}^+$ to $\mathbb{D}$ are in the form of

$$w = \mathrm{e}^{\mathrm{i}\theta} \frac{z - a}{z - \overline{a}}, \quad a \in \mathbb{H}^+, \theta \in \mathbb{R}. \tag{5.1.2}$$

Proof: First assume $y = \mathfrak{Im}(z) > 0$. It follows that

$$|w| = \left| \frac{z - a}{z - \overline{a}} \right| = \sqrt{\frac{(x - \mathfrak{Re}(a))^2 + (y - \mathfrak{Im}(a))^2}{(x - \mathfrak{Re}(a))^2 + (y + \mathfrak{Im}(a))^2}} < 1.$$

Therefore, this transformation maps $\mathbb{H}^+$ to $\mathbb{D}$. The inverse mapping is equal to

$$z = \frac{w\overline{a} - a\mathrm{e}^{\mathrm{i}\theta}}{w - \mathrm{e}^{\mathrm{i}\theta}}. \tag{5.1.3}$$

Assume $w \in \mathbb{D}$. We then have

$$\begin{aligned}\mathfrak{Im}(z) &= \mathfrak{Im}\left(\frac{\left(w\overline{a}-a\mathrm{e}^{\mathrm{i}\theta}\right)\left(\overline{w}-\mathrm{e}^{-\mathrm{i}\theta}\right)}{\left|w-\mathrm{e}^{\mathrm{i}\theta}\right|^2}\right)\\ &= \frac{|w|^2\,\mathfrak{Im}(\overline{a})-\mathfrak{Im}\left(a\mathrm{e}^{\mathrm{i}\theta}\overline{w}\right)-\mathfrak{Im}\left(w\overline{a}\mathrm{e}^{-\mathrm{i}\theta}\right)+\mathfrak{Im}(a)}{\left|w-\mathrm{e}^{\mathrm{i}\theta}\right|^2}\\ &= \frac{\left(1-|w|^2\right)\mathfrak{Im}(a)}{\left|w-\mathrm{e}^{\mathrm{i}\theta}\right|^2} > 0.\end{aligned}$$

Hence, z maps $\mathbb{D}$ to $\mathbb{H}^+$ univalently and surjectively since it is also an element in $\operatorname{Aut}\left(\hat{\mathbb{C}}\right)$.

Let $\psi(z)$ be the biholomorphism from $\mathbb{H}^+$ to $\mathbb{D}$ in the form of $\psi(z)=\frac{z-\mathrm{i}}{z+\mathrm{i}}$ (for $\theta=0$ and $a=\mathrm{i}$, known as the *Cayley transform*). Let f be an arbitrary biholomorphism from $\mathbb{H}^+$ to $\mathbb{D}$. It follows that $\varphi=f\circ\psi^{-1}$ is a holomorphic automorphism on $\mathbb{D}$. Since $\varphi\in\operatorname{Aut}(\mathbb{D})$, we have

$$\begin{aligned}f(z)=\varphi\circ\psi(z) &= \mathrm{e}^{\mathrm{i}\theta}\frac{z(1-a)-\mathrm{i}(a+1)}{z(1-\overline{a})+\mathrm{i}(\overline{a}+1)}\\ &= \mathrm{e}^{\mathrm{i}\theta}\frac{z\frac{1-a}{1-\overline{a}}+\mathrm{i}\frac{a+1}{\overline{a}-1}}{z-\mathrm{i}\frac{\overline{a}+1}{\overline{a}-1}}\\ &= \mathrm{e}^{\mathrm{i}\theta}\frac{1-a}{1-\overline{a}}\frac{z-\mathrm{i}\frac{a+1}{1-a}}{z-\overline{\mathrm{i}\frac{a+1}{1-a}}}.\end{aligned}$$

Obviously, $\mathrm{e}^{\mathrm{i}\theta}\frac{1-a}{1-\overline{a}}$ attains every value on the unit disk for varying a and θ. Similarly, the values attained by $\mathrm{i}\frac{a+1}{1-a}$ cover the upper half-plane for $a\in\mathbb{D}$ (since it is in the form of (5.1.3)). Thus, all biholomorphisms from $\mathbb{H}^+$ to $\mathbb{D}$ are in the form of (5.1.2). ☐

Let us now introduce some important properties of linear fractional transformations. By Proposition 4.4.2.1, it follows that the composition of two linear fractional transformations is also a linear fractional transformation.

Theorem 5.1.4: Let $\mathcal{C}$ be the collection of subsets of $\hat{\mathbb{C}}$ that are circles or $L\cup\{\infty\}$, where L is a straight line in $\mathbb{C}$ (known as *generalized circles*). Then every linear fractional transformation $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ maps elements of $\mathcal{C}$ to elements of $\mathcal{C}$.

Proof: Since each linear fractional transformation is a composition of maps in the form of $z\mapsto az$, $z\mapsto z+b$, and $z\mapsto\frac{1}{z}$, it suffices to show that these maps preserve the property of being a circle or a straight line. Consider a circle defined implicitly with

$$\alpha(x^2+y^2)+\beta x+\gamma y+\delta=0,\quad x,y\in\mathbb{R},\alpha,\beta,\gamma,\delta\in\mathbb{R}.$$

For $z=x+\mathrm{i}y$, this can be rewritten as

$$\begin{aligned}&\alpha z\overline{z}+\beta\frac{z+\overline{z}}{2}+\gamma\frac{z-\overline{z}}{2\mathrm{i}}+\delta\\ &=\alpha z\overline{z}+\xi z+\overline{\xi}\overline{z}+\delta=0\quad\text{for}\quad\xi=\frac{\beta}{2}+\frac{\gamma}{2\mathrm{i}}.\end{aligned}\tag{5.1.4}$$

If $\alpha=0$, the equation represents a straight line. It is easy to see that a complex dilation or a translation of z will preserve the property of being a straight line or a circle. Indeed, by letting $z=a\zeta$ for nonzero a in (5.1.4), we have

$$\alpha|a|^2\zeta\overline{\zeta}+\xi a\zeta+\overline{\xi}\overline{a}\overline{\zeta}+\delta=0,$$

which is trivially in the form of (5.1.4). Similarly, if we substitute $z=\zeta+b$, we have

$$\begin{gathered}\alpha(\zeta+b)\left(\overline{\zeta}+\overline{b}\right)+\xi(\zeta+b)+\overline{\xi}\left(\overline{\zeta}+\overline{b}\right)+\delta=0\\ \alpha\zeta\overline{\zeta}+\left(\xi+\alpha\overline{b}\right)\zeta+\left(\overline{\xi}+\alpha b\right)\overline{\zeta}+\alpha|b|^2+2\,\mathfrak{Re}(\xi b)+\delta=0.\end{gathered}$$

If we substitute $z=\frac{1}{\zeta}$, we have

$$\delta\zeta\overline{\zeta}+\xi\overline{\zeta}+\overline{\xi}\zeta+\alpha=0,$$

which is in the form of (5.1.4). □

Remark: As in Example 5.1.2, we can consider extended straight lines in the form of $L\cup\{\infty\}$ as generalized circles in the Riemann sphere. In other words, the extended line can be geometrically visualized by a circle with infinite radius. In fact, when a circle on the Riemann sphere is projected stereographically onto the complex plane, the result is always either a circle or a straight line.

Definition 5.1.1 *(Cross-Ratio)*: Let $z_1,z_2,z_3,z_4\in\hat{\mathbb{C}}$ be points such that at least three of them are distinct. The *cross-ratio* of these points is defined as

$$(z_1,z_2;z_3,z_4)=\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}.$$

If at least one of the four points is ∞, then the cross-ratio is defined by the limit:

$$(\infty, z_2; z_3, z_4) = \frac{z_2 - z_4}{z_2 - z_3}, (z_1, \infty; z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4}$$
$$(z_1, z_2; \infty, z_4) = \frac{z_2 - z_4}{z_1 - z_4}, (z_1, z_2; z_3, \infty) = \frac{z_1 - z_3}{z_2 - z_3}$$

One important property of the cross-ratio is that it is invariant under linear fractional transformations. In other words, if f is a linear fractional transformation, then

$$(f(z_1), f(z_2); f(z_3), f(z_4)) = (z_1, z_2; z_3, z_4).$$

The proof is trivial and can be verified by substituting the definition of the linear fractional transformation into the definition of the cross-ratio.

Furthermore, if a function $f(z_1, z_2, z_3, z_4)$ is invariant under the group of linear fractional transformations, then it is a function of the cross-ratio. In other words, the cross-ratio is the only invariant under the group of linear fractional transformations $\mathrm{Aut}\left(\hat{\mathbb{C}}\right)$. Indeed, suppose that

$$f(\varphi(z_1), \varphi(z_2), \varphi(z_3), \varphi(z_4)) = f(z_1, z_2, z_3, z_4).$$

We aim to show that f is a function of a cross-ratio. Let

$$\varphi(z) = \frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)}$$

be a linear fractional transformation. Then we have

$$f(\varphi(z_1), \varphi(z_2), \varphi(z_3), \varphi(z_4)) = f((z_1, z_2; z_3, z_4), 1, 0, \infty),$$

which is a function of the cross-ratio.

5.2 Normal Families

A collection of functions is better known as a *family* of functions. One important distinguishing property of families of functions, as opposed to sequences, is that families may be uncountable and may not be indexed by the natural numbers. We will now introduce the following classification of families of functions:

Definition 5.2.1 *(Normal Family)*: A family of holomorphic functions $\mathcal{F}$ defined on a region $U \subseteq \mathbb{C}$ is said to be *normal* if every sequence of functions in $\mathcal{F}$ has a locally uniformly (compactly) convergent subsequence on U.

The following notion was introduced and formalized by the Italian mathematicians Cesare Arzelà and Giulio Ascoli to formulate a clear distinction in how uniformity is applied.

Definition 5.2.2 *(Equicontinuity)*: A family of functions $\mathcal{F}$ defined on a region $U \subseteq \mathbb{C}$ is said to be *equicontinuous* at a point $z_0 \in U$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ (that may depend on z_0) such that for all $f \in \mathcal{F}$ and all $z \in U$ with $|z - z_0| < \delta$, we have $|f(z) - f(z_0)| < \varepsilon$.

In contrast, the uniform continuity of a function f guarantees that δ may be chosen independently of z_0. In the case of (pointwise) equicontinuity, it is chosen independently of $f \in \mathcal{F}$. A family of functions is said to be *uniformly equicontinuous* on U if δ can be chosen independently of both z_0 and $f \in \mathcal{F}$ (in other words, it attains a positive infimum in U). Similar to Theorem 1.2.15

Theorem 5.2.1: A family of functions $\mathcal{F}$ that is pointwise equicontinuous on every point $z \in K \subset \mathbb{C}$ for a compact set K is uniformly equicontinuous on K.

Proof: Fix $z \in K$. By pointwise equicontinuity, $\forall \varepsilon > 0$, $\exists \delta_z > 0$ such that $\forall f \in \mathcal{F}, \forall \zeta \in D(z, \delta_z) \cap K$,

$$|f(\zeta) - f(z)| < \frac{\varepsilon}{2}. \tag{5.2.1}$$

The collection $\left\{D\left(z, \frac{\delta_z}{2}\right)\right\}_{z \in K}$ forms an open cover of K, and by the Heine–Borel Theorem, it admits a finite subcover $\left\{D\left(z_k, \frac{\delta_{z_k}}{2}\right)\right\}_{k \in \mathbb{N}_{\leq n}}$ for some finite $n \in \mathbb{N}$. Let $\delta = \min_{k \in \mathbb{N}_{\leq n}} \left(\frac{\delta_{z_k}}{2}\right)$.

For any $z, w \in K$ such that $|z - w| < \delta$, $\exists j \in \mathbb{N}_{\leq n}$ such that $z \in D\left(z_j, \frac{\delta_{z_j}}{2}\right)$. Evidently,

$$|z_j - w| \leq |z_j - z| + |z - w| < \frac{\delta_{z_j}}{2} + \delta \leq \delta_{z_j}.$$

Therefore, from (5.2.1), we have $\forall f \in \mathcal{F}$,

$$|f(z_j) - f(w)| < \frac{\varepsilon}{2}, \quad |f(z_j) - f(z)| < \frac{\varepsilon}{2}.$$

Hence, $\forall f \in \mathcal{F}$, we have

$$|f(z) - f(w)| \leq |f(w) - f(z_j)| + |f(z_j) - f(z)| < \varepsilon,$$

which proves the uniform equicontinuity of $\mathcal{F}$. ☐

The following theorem is important in many areas of mathematical analysis and has a plethora of generalizations. It was first introduced by Ascoli (who proved the sufficiency of compactness) and later formalized by Arzelà, who proved the necessity of uniform equicontinuity and uniform boundedness.

Theorem 5.2.2 *(Arzelà–Ascoli)*: Let $\mathcal{F}$ be a family of complex continuous functions defined on a compact subset $K \subseteq \mathbb{C}$. Then, $\mathcal{F}$ is uniformly bounded and uniformly equicontinuous on K iff $\mathcal{F}$ is normal on K.

Proof: We will first prove the sufficiency of uniform boundedness and uniform equicontinuity. Let $\{f_n\}_{n\in\mathbb{N}}$ be any sequence in $\mathcal{F}$. By the uniform boundedness of $\mathcal{F}$, there exists a constant $M > 0$ such that $|f_n(z)| \le M$ for all $z \in K$ and all $n \in \mathbb{N}$.

Let $\{\zeta_k\}_{k\in\mathbb{N}}$ be a countably dense subset of K. By the Bolzano–Weierstrass Theorem (Theorem 1.1.2), there exists a subsequence of $\{f_n\}_{n\in\mathbb{N}}$, namely $\left\{f_{n_{1,j}}\right\}_{j\in\mathbb{N}}$, such that $\left\{f_{n_{1,j}}(\zeta_1)\right\}_{j\in\mathbb{N}}$ is convergent. The set $\left\{f_{n_{1,j}}(\zeta_2)\right\}_{j\in\mathbb{N}}$ is also bounded by M, and hence, by the Bolzano–Weierstrass Theorem, it too has a convergent subsequence $\left\{f_{n_{2,j}}(\zeta_2)\right\}_{j\in\mathbb{N}}$. Similarly, there exists a subsequence of $\left\{f_{n_{2,j}}\right\}_{j\in\mathbb{N}}$, namely $\left\{f_{n_{3,j}}\right\}_{j\in\mathbb{N}}$, such that $\left\{f_{n_{3,j}}(\zeta_3)\right\}_{j\in\mathbb{N}}$ is convergent.

By the method of construction, we have:

$$\begin{gathered} n_{1,1} < n_{1,2} < \cdots < n_{1,j} < \cdots \\ n_{2,1} < n_{2,2} < \cdots < n_{2,j} < \cdots \\ \vdots \\ n_{k,1} < n_{k,2} < \cdots < n_{k,j} < \cdots \\ \ddots, \end{gathered} \tag{5.2.2}$$

and furthermore, the sequence in each row is a subsequence of the previous row. As a result, we have

$$\begin{gathered} n_{1,1} \le n_{2,1} \le \cdots \le n_{k,1} \le \cdots \\ n_{1,2} \le n_{2,2} \le \cdots \le n_{k,2} \le \cdots \\ \vdots \\ n_{1,j} \le n_{2,j} \le \cdots \le n_{k,j} \le \cdots \\ \ddots. \end{gathered} \tag{5.2.3}$$

We will now invoke a diagonalization argument. Since the sequences above in (5.2.2) are strictly increasing and from the results of (5.2.3), it follows that $\left\{n_{j,j}\right\}_{j\in\mathbb{N}}$ is strictly increasing. Let $n_{j,j}$ be denoted by n_j'. Since $\mathcal{F}$ is uniformly equicontinuous on K, $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that $\forall z, z' \in K$ satisfying $|z - z'| < \delta$, $\forall j \in \mathbb{N}$, we have

$$\left|f_{n_j'}(z) - f_{n_j'}(z')\right| < \frac{\varepsilon}{3}. \tag{5.2.4}$$

Since each $\left\{f_{n_{k,j}}\right\}_{j\in\mathbb{N}}$ is convergent at ζ_k (for a fixed k) by construction, and since $\left\{n_j'\right\}_{j\geq k}$ is a subsequence of $\left\{n_{k,j}\right\}_{j\in\mathbb{N}}$, it is evident that $\left\{f_{n_j'}\right\}_{j\in\mathbb{N}}$ is convergent at each ζ_k. We then have that $\forall k\in\mathbb{N}$, $\exists N=N(\varepsilon,k)\in\mathbb{N}$ such that $\forall i,j>N$,

$$\left|f_{n_i'}(\zeta_k)-f_{n_j'}(\zeta_k)\right|<\frac{\varepsilon}{3}.$$

For the fixed value of ε, the collection $\{D(\zeta_k,\delta)\}_{k\in\mathbb{N}}$ forms an open cover of K, and by the Heine–Borel Theorem (Theorem 1.1.3), it admits finite subcovering $\{D(\zeta_k,\delta)\}_{k\in\{1,\ldots,l\}}$ for some finite $l=l(\varepsilon)\in\mathbb{N}$.

Hence, $\exists k=k(\varepsilon)\leq l$ such that any point $z\in K$ lies in $D(\zeta_k,\delta)$. By (5.2.4), we have that

$$\left|f_{n_j'}(z)-f_{n_j'}(\zeta_k)\right|<\frac{\varepsilon}{3},\quad\left|f_{n_i'}(z)-f_{n_i'}(\zeta_k)\right|<\frac{\varepsilon}{3}.$$

Letting $\tilde{N}=\tilde{N}(\varepsilon)=\max(\{N(\varepsilon,1),\ldots,N(\varepsilon,l(\varepsilon))\})$, we have that $\forall i,j>\tilde{N}$, $\forall z\in K$,

$$\begin{aligned}\left|f_{n_j'}(z)-f_{n_i'}(z)\right|&\leq\left|f_{n_j'}(z)-f_{n_j'}(\zeta_k)\right|+\left|f_{n_j'}(\zeta_k)-f_{n_i'}(\zeta_k)\right|\\&\quad+\left|f_{n_i'}(\zeta_k)-f_{n_i'}(z)\right|\\&=\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

Hence, the sequence is uniformly convergent on K by the Cauchy Criterion (Theorem 2.3.1).

For the proof of the necessity, we will first assume the normality of $\mathcal{F}$ in K.

For the sake of contradiction, assume that $\mathcal{F}$ is not uniformly bounded. Then $\forall n\in\mathbb{N}$, $\exists f_n\in\mathcal{F}$ and $\exists z_n\in K$ such that $|f_n(z_n)|>n$. By assumption, this sequence has a subsequence $\left\{f_{n_k}\right\}_{k\in\mathbb{N}}$ that uniformly converges. Hence, $\exists N\in\mathbb{N}$ such that $\forall k>N$, $\forall z\in K$, $\left|f_{n_k}(z)-f(z)\right|<1$. By the reverse triangle inequality, it follows that $\left|f_{n_k}(z)\right|<|f(z)|+1$. Since f is continuous on K by Theorem 2.3.5, it is bounded by some M_1 (Theorem 1.2.13). Let $M_2=\max_{k\in\mathbb{N}_{\leq N}}\sup_{z\in K}\left|f_{n_k}(z)\right|$. It follows that this subsequence is uniformly bounded by $\max(\{M_1+1,M_2\})$. However, since $\left|f_{n_k}\left(z_{n_k}\right)\right|>n_k\to\infty$ for any k, this subsequence cannot be uniformly bounded, and hence we have a contradiction.

We will now assume that $\mathcal{F}$ is not pointwise equicontinuous at some arbitrary point $z_0 \in K$. In other words, $\exists \varepsilon > 0$ such that $\forall \delta > 0, \exists f \in \mathcal{F}, \exists z \in K$ such that $|z - z_0| < \delta$ satisfying

$$|f(z) - f(z_0)| > \varepsilon.$$

Let us define sequences $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $\{z_n\}_{n\in\mathbb{N}} \subseteq K$ such that $|z_n - z_0| < \frac{1}{n}$ and

$$|f_n(z_n) - f_n(z_0)| > \varepsilon.$$

Since $\mathcal{F}$ is assumed to be normal, the sequence $\{f_n\}_{n\in\mathbb{N}}$ has a uniformly convergent subsequence $\left\{f_{n_k}\right\}_{k\in\mathbb{N}}$ converging to a continuous function f. In particular, since uniform convergence preserves continuity (Theorem 2.3.5), the limit f is continuous at z_0, and hence,

$$f_{n_k}(z_0) - f(z_0) \to 0, \quad f(z_0) - f\left(z_{n_k}\right) \to 0, \quad f_{n_k}\left(z_{n_k}\right) - f\left(z_{n_k}\right) \to 0,$$

where the rightmost inequality is derived from the fact that $f_{n_k} \rightrightarrows f$ on K. Thus,

$$\left|f_{n_k}\left(z_{n_k}\right) - f_{n_k}(z_0)\right| \to 0,$$

which contradicts the result that $\left|f_{n_k}\left(z_{n_k}\right) - f_{n_k}(z_0)\right| > \varepsilon$ for all k.

Hence, by contradiction, $\mathcal{F}$ is pointwise equicontinuous on all of K. By Theorem 5.2.1, $\mathcal{F}$ must be uniformly equicontinuous on K. ☐

The notions and results introduced have profound implications and uses in the theory of differential equations and harmonic analysis.

In the definition of equicontinuity used in the Arzelà–Ascoli theorem, the distance is taken with respect to the Euclidean metric. However, the theorem continues to hold for other metrics as well, with the proof requiring little modification. We will rely on this formulation in Section 8.4.

Lastly, we will prove Montel's Theorem in preparation of the Riemann Mapping Theorem (Theorem 5.3.1).

Definition 5.2.3: Let $\mathcal{F}$ be a family of functions defined on an open set $U \subseteq \mathbb{C}$. The family $\mathcal{F}$ is said to be *locally uniformly bounded* if, for every point $z \in U$, there exists a neighborhood $V \subseteq U$ of z such that $\mathcal{F}$ is uniformly bounded on V. This condition is equivalent to the condition that $\mathcal{F}$ is uniformly bounded on all compact subsets K of U.

Obviously, the equivalence is established similarly to local finiteness and locally uniform convergence.

Theorem 5.2.3 *(Montel's Theorem)*: Let $U \subseteq \mathbb{C}$ be open, and suppose that $\mathcal{F}$ is a family of holomorphic functions on U. Then, $\mathcal{F}$ is locally uniformly bounded on U iff $\mathcal{F}$ is a normal family.

Proof: Obviously, if $\mathcal{F}$ is normal on U, for any compact $K \subset U$, it follows that $\mathcal{F}$ is normal on K, and the uniform boundedness on K follows from the Arzelà–Ascoli Theorem (Theorem 5.2.2).

Conversely, we will first assume that $\mathcal{F}$ is locally uniformly bounded. Let $z \in U$ be arbitrary, and choose $R_z > 0$ such that $\overline{D(z, R_z)} \subset U$. Therefore, it follows that $\mathbb{C} \setminus U$ is closed and disjoint from $\overline{D(z, R_z)}$ and the distance between them is positive. Let this distance be

$$d_z = \inf\left(\left\{|\zeta - \zeta'| : \zeta \in \mathbb{C} \setminus U, \zeta' \in \overline{D(z, R_z)}\right\}\right).$$

It follows that the disk $V_z = D\left(z, R_z + \frac{d_z}{2}\right)$ is relatively compact in U. By Corollary 3.2.5.1, there exists a finite constant $c_z' > 0$ independent of $f \in \mathcal{F}$ such that

$$|f'(\zeta)| < c_z' \max_{\substack{\xi \in \overline{V_z} \\ \tilde{f} \in \mathcal{F}}} \left|\tilde{f}(\xi)\right|, \quad \forall \zeta \in \overline{D(z, R_z)}, \forall f \in \mathcal{F}$$

where the maximum on the right-hand side is finite by assumption of the locally uniform boundedness of $\mathcal{F}$. For simplicity, let

$$c_z = c_z' \max_{\substack{\xi \in \overline{V_z} \\ \tilde{f} \in \mathcal{F}}} \left|\tilde{f}(\xi)\right|.$$

Let $\xi, \xi' \in \overline{D(z, R_z)}$ be arbitrary and distinct, and let γ be the straight curve from ξ to ξ'. For an arbitrary function $f \in \mathcal{F}$, we have that

$$|f(\xi') - f(\xi)| = \left|\int_\gamma f'(\zeta)\,\mathrm{d}\zeta\right| \leq c_z \int_\gamma |\mathrm{d}z| = c_z|\xi' - \xi|.$$

Therefore, $\mathcal{F}$ is uniformly equicontinuous in $\overline{D(z, R_z)}$ (and also in $D(z, R_z)$). Indeed, $\forall \varepsilon > 0$, we can choose $\delta_z = \frac{\varepsilon}{c_z}$ and the assertion follows.

Let $K \subset U$ be compact and arbitrary. The collection $\{D(z, R_z)\}_{z \in K}$ forms an open cover of K and by the Heine–Borel Theorem (Theorem 1.1.3) admits a finite subcover $\left\{D\left(z_k, R_{z_k}\right)\right\}_{k \in \mathbb{N}_{\leq n}}$ for some finite $n \in \mathbb{N}$. If we let $\delta = \min_{k \in \mathbb{N}_{\leq n}}(\delta_k)$, it follows that $\mathcal{F}$ is uniformly equicontinuous on K. By the Arzelà–Ascoli Theorem (Theorem 5.2.2), any sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}$ has a uniformly convergent subsequence $\left\{f_{n_k}\right\}_{k \in \mathbb{N}}$ on K.

Let $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ be arbitrary. Let U be exhausted by the compact sets $\{K_n\}_{n\in\mathbb{N}}$. By the argument above, we may extract a subsequence $\left\{f_{n_{1,j}}\right\}_{j\in\mathbb{N}} \subseteq \{f_n\}_{n\in\mathbb{N}}$ that uniformly converges on K_1. By the same argument, there exists a subsequence $\left\{f_{n_{2,j}}\right\}_{j\in\mathbb{N}} \subseteq \left\{f_{n_{1,j}}\right\}_{j\in\mathbb{N}}$ that uniformly converges on K_2. Let $n'_j = n_{j,j}$.

We will now invoke the same diagonalization argument as in the proof of the Arzelà–Ascoli Theorem (Theorem 5.2.2). Let $K \subset U$ be an arbitrary compact set. It follows that for some $k \in \mathbb{N}$, $K_k \supseteq K$. Since $\left\{f_{n'_j}\right\}_{j\geq k} \subseteq \left\{f_{n_{k,j}}\right\}_{j\in\mathbb{N}}$ is the subsequence of a sequence that converges on K, the assertion follows. □

5.3 The Riemann Mapping Theorem

The Riemann Mapping Theorem is one of the most profound results in complex analysis; in the case of one dimension, it establishes sufficient conditions for the biholomorphic equivalence between two open subsets of the complex plane.

If there exists a biholomorphism f between two regions, then the two regions are said to be *conformally equivalent*, *holomorphically equivalent*, or *biholomorphically equivalent*. As a required intermediate for the proof, we first introduce:

Definition 5.3.1 *(Holomorphic Logarithms)*: Suppose $\Phi : U \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is holomorphic, where U is simply connected. Define the *holomorphic logarithm* of $\Phi(z)$ to be a branch of

$$\log(\Phi(z)) = \int_\gamma \frac{\Phi'(\zeta)}{\Phi(\zeta)}\,\mathrm{d}\zeta + \log(\Phi(z_0))$$

for any $z_0 \in U$, where the integral is path-independent $\gamma \subset U$ is any piecewise C^1 curve from z_0 to z.

Similarly, define the *holomorphic powers* of $\Phi(z)$ to be branches of $(\Phi^\alpha)(z) = \mathrm{e}^{\alpha \log(\Phi(z))}$, where $\log(\Phi(z))$ is the holomorphic logarithm.

The path independence of the definition is provided by the simple connectivity of U. The result is the heuristic concatenation of several different branches of the complex logarithm, unique up to an additive factor of $2\pi\mathrm{i}k$, where this additive factor is the same throughout.

Theorem 5.3.1 *(Riemann Mapping Theorem)*: Let $U \subset \mathbb{C}$ (a proper subset, in other words, $U \neq \mathbb{C}$) be a simply connected (nonempty) open region. Let $z_0 \in U$ be arbitrary. Then there exists a unique biholomorphism $f : U \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) \in \mathbb{R}_{>0}$.

Proof: First consider the case for when U is a bounded region. In other words, $\exists R > 0$ such that $U \subseteq D(0, R)$.

Define $\mathcal{F}$ to be the family of all univalent functions $\alpha : U \to \mathbb{D}$ (not necessarily surjective) such that $\alpha(z_0) = 0$. This family is well-defined and nonempty.

To prove this assertion, observe that since $z_0 \in D(0, R)$, it follows that $\forall z \in U \subseteq D(0, R)$, $|z - z_0| < 2R$, and consequently, $\left|\frac{z-z_0}{2R}\right| < 1$. Therefore,

$$\alpha(z) = \frac{z - z_0}{2R}$$

maps U to $\mathbb{D}$, and it is linear and univalent. This shows that $\alpha \in \mathcal{F}$. It is easy to prove that $\mathcal{F}$ is infinite; any function in the form of $z \mapsto \frac{z-z_0}{\zeta}$ for $\zeta \geq 2R$ also lies in $\mathcal{F}$.

Since $\mathcal{F}$ is uniformly bounded on U, by Montel's Theorem (Theorem 5.2.3), $\mathcal{F}$ is a normal family. Let $r > 0$ satisfy $\overline{D(z_0, r)} \subset U$. Then by Cauchy's Estimate (Theorem 3.2.2), $\forall \alpha \in \mathcal{F}$, $|\alpha'| \leq \frac{1}{r}$ on $\overline{D(z_0, r)}$. Hence, we have

$$0 < M = \sup_{\alpha \in \mathcal{F}} |\alpha'(z_0)| \leq \frac{1}{r}, \tag{5.3.1}$$

where we can assure that M is positive since each $\alpha \in \mathcal{F}$ is univalent at z_0 and by Lemma 5.1.1.

If M is an accumulation point of $\{|\alpha'(z_0)|\}_{\alpha \in \mathcal{F}}$, there exists a sequence $\{\alpha_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}$ such that $\{|\alpha_n'(z_0)|\}_{n \in \mathbb{N}}$ converges to M. If M is attained as a maximum or that $|\alpha'(z_0)| = M$ for some $\alpha \in \mathcal{F}$, we may let each $\alpha_n \equiv \alpha$.

By the normality of $\mathcal{F}$, there exists a subsequence $\left\{\alpha_{n_k}(z)\right\}_{k \in \mathbb{N}} \subseteq \{\alpha_n(z)\}_{n \in \mathbb{N}}$ such that $\left\{\alpha_{n_k}(z)\right\}_{k \in \mathbb{N}}$ is locally uniformly convergent in U to a function $\tilde{\tilde{\alpha}}(z)$ (holomorphy of which is provided by Theorem 4.1.1). By definition, $\left|\tilde{\tilde{\alpha}}'(z_0)\right| = M$, and define a function sequence with $\tilde{\alpha}_{n_k} = \alpha_{n_k} \frac{|\tilde{\tilde{\alpha}}'(z_0)|}{\tilde{\tilde{\alpha}}'(z_0)} \in \mathcal{F}$, whose locally uniform limit is f. It follows that f is a rotation of $\tilde{\tilde{\alpha}}$ such that $f'(z_0) = M$.

Let $\zeta_1, \zeta_2 \in U$ be arbitrary and different. Choose $r' > 0$ to satisfy $0 < r' < |\zeta_1 - \zeta_2|$, and let $\psi_k(z) = \tilde{\alpha}_{n_k}(z) - \tilde{\alpha}_{n_k}(\zeta_2)$. Since each $\tilde{\alpha}_{n_k}$ is univalent in U, it follows that each ψ_k is non-vanishing in $U \setminus \{\zeta_2\}$ and consequently, in $\overline{D(\zeta_1, r')}$. By Theorem 3.3.5, it follows that the locally uniform limit of ψ_k, or $\psi = f(z) - f(\zeta_2)$, is either non-vanishing or is identically zero in $\overline{D(\zeta_1, r')}$. The latter is an impossibility since $\psi'(z_0) = M > 0$. Hence, $f(z) = f(\zeta_2)$ has no solutions for $z \in \overline{D(\zeta_1, r')}$. In particular, $f(\zeta_1) \neq f(\zeta_2)$. By the arbitrariness of ζ_1 and ζ_2, the univalence of f follows.

Additionally, since $\forall k \in \mathbb{N}$, $\left|\tilde{\alpha}_{n_k}\right| < 1$ in U, it follows that $f(U) \subseteq \overline{\mathbb{D}}$. By the Open Mapping Theorem (Theorem 5.1.1), the condition becomes $f(U) \subseteq \mathbb{D}$. Since $\tilde{\alpha}_{n_k}(z_0) = 0$ for all $k \in \mathbb{N}$ and $\tilde{\alpha}_{n_k}(z_0) \to 0 = f(z_0)$, it follows that $f \in \mathcal{F}$.

Lastly, we aim to prove that f maps U to $\mathbb{D}$ surjectively. For the sake of contradiction, assume that $\exists \xi \in \mathbb{D}^*$ such that $\xi \notin f(U)$. Consider the unit disk automorphism $\varphi_\xi(z) = \frac{z-\xi}{1-z\overline{\xi}}$. Since $\varphi_\xi(z)$ vanishes when $z = \xi$, and since $f(z) = \xi$ has no solutions in U, there exists a holomorphic square root

$$\mu(z) = \sqrt{\varphi_\xi \circ f(z)} \in \mathbb{D}$$

for $z \in U$. Let $\tau = \mu(z_0) = \sqrt{-\xi}$, and let

$$\eta(z) = \varphi_\tau \circ \mu(z),$$

where $\varphi_\tau = \frac{z-\tau}{1-z\overline{\tau}}$. Since $\eta(z_0) = \varphi_\tau(\tau) = 0$, it follows that $\eta \in \mathcal{F}$. Let $\tilde{\eta} = \frac{|\eta'(z_0)|}{\eta'(z_0)}\eta$, which is also in $\mathcal{F}$. However, since $\tilde{\eta}' = \frac{|\eta'(z_0)|}{\eta'(z_0)}\eta'$, we have

$$\begin{aligned}
\tilde{\eta}'(z_0) &= \left| \frac{f'(z_0)(\varphi_\tau' \circ \tau)\left(\varphi_\xi' \circ 0\right)}{2\sqrt{\varphi_\xi \circ 0}} \right| \\
&= \frac{M}{2\sqrt{|\xi|}} \frac{1 - \overline{\tau}\tau}{(1-\tau\overline{\tau})^2} \frac{1 - \overline{\xi}\xi}{\left(1 - 0\overline{\xi}\right)^2} \\
&= \frac{M}{2\sqrt{|\xi|}} \frac{1 - |\xi|^2}{1 - |\xi|} = \frac{M(1 + |\xi|)}{2\sqrt{|\xi|}}.
\end{aligned}$$

Additionally, since

$$\left(\sqrt{|\xi|} - 1\right)^2 > 0 \Longleftrightarrow 1 + |\xi| > 2\sqrt{|\xi|} \Longleftrightarrow \frac{1 + |\xi|}{2\sqrt{|\xi|}} > 1,$$

it follows that $\tilde{\eta}'(z_0) > M$, which is a contradiction of (5.3.1).

Hence, $f : U \to \mathbb{D}$ is biholomorphic. To prove the uniqueness of f, suppose $g : U \to \mathbb{D}$ is an arbitrary biholomorphism such that $g(z_0) = 0$ and $g'(z_0) > 0$. Then, $\varphi = f \circ g^{-1} \in \operatorname{Aut}(\mathbb{D})$, and by Theorem 3.5.1, $\varphi(z) = \varphi_a\left(z\mathrm{e}^{\mathrm{i}\theta}\right)$ for some $a \in \mathbb{D}$ and $0 \leq \theta < 2\pi$. Since $\varphi(0) = 0$, it follows that $a = 0$. Since $\varphi'(0) = f'(z_0)\left(g^{-1}\right)'(0) = f' \frac{z_0}{g'(z_0)} > 0$, and $\varphi'(0) = \varphi_0'(0)\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}\theta} > 0$, it follows that $\theta = 0$. Hence, we have $\varphi(z) = z$ and $f \equiv g$.

Next, assume that U is unbounded. It is easy to show that the boundary ∂U contains at least two points. Indeed, if $\partial U = \varnothing$, U would be closed because $\partial U \subseteq U$ and open by assumption. By Theorem 3.2.1.3, U would either be equal to $\varnothing$ or $\mathbb{C}$, both of which are impossibilities. Additionally, if ∂U comprises exactly one point $a \in \mathbb{C}$, then in subspace defined by $X = \mathbb{C} \setminus \{a\}$, U is clopen (by the same reason as before, open by assumption and closed because $X \setminus U = \mathbb{C} \setminus \overline{U}$ is open). It follows that $U = X = \mathbb{C} \setminus \{a\}$, which is not simply connected.

Suppose ξ_1 and ξ_2 are two distinct points in ∂U. Let us apply the linear transformation $\rho(z) = \frac{z-\xi_1}{\xi_2-\xi_1}$ to U, and denote the resulting region by $U' = \rho(U)$. It follows that $0, 1 \in \partial U'$. Consider a branch $\psi(z)$ of the holomorphic square root $z \mapsto \sqrt{z-1}$ (existent by the holomorphic logarithm from simple connectivity and the fact that $1 \notin U'$). Trivially, ψ is univalent in U'.

In addition, we assert that $\psi(U') \cap (-\psi(U')) = \varnothing$. If not, then $\exists \xi \in \psi(U')$ such that $-\xi \in \psi(U')$. By definition, $\exists z_1, z_2 \in U'$ such that $\psi(z_1) = \xi$ and $\psi(z_2) = -\xi$. It would then follow that $\sqrt{z_1 - 1} = -\sqrt{z_2 - 1}$ and $z_1 = z_2$. It follows that $\xi = 0$, which is obtained when $z_1 = z_2 = \psi^{-1}(\xi) = 1$. Since $1 \in \partial U'$ and U' is open, $1 \notin U'$, which is an impossibility.

Fix $\xi \in \psi(U')$ to be arbitrary. By the Open Mapping Theorem (Theorem 5.1.1), there exists an open neighborhood $D(\xi, \varepsilon) \subseteq \psi(U')$. It follows that $D(-\xi, \varepsilon) \cap \psi(U') = \varnothing$. Therefore, $\forall z \in U'$, $|\psi(z) + \xi| \geq \varepsilon$, and consequently, $\left|\frac{1}{\psi(z)+\xi}\right| \leq \frac{1}{\varepsilon}$. Hence, the function $\varphi(z) = \frac{1}{z+\xi}$ maps U' to a bounded region that lies within the compact disk $\overline{D\left(0, \frac{1}{\varepsilon}\right)}$. Denote $\varphi \circ \psi(U')$ by $\tilde{U}$.

It is easy to see that $\tilde{U}$ is simply connected. Let $\tilde{U} = \varphi \circ \psi \circ \rho(U)$. To prove this, it suffices to show that the line integral of any holomorphic function over any closed curve in $\tilde{U}$ vanishes. Let $g : \tilde{U} \to \mathbb{C}$ be holomorphic, and let $\Gamma \subset \tilde{U}$ be a closed piecewise C^1 curve. Then

$$\oint_{\Gamma} g(w)\,\mathrm{d}w = \oint_{\rho^{-1}\circ\psi^{-1}\circ\varphi^{-1}(\Gamma)} g \circ \varphi \circ \psi \circ \rho(z) \cdot (\varphi \circ \psi \circ \rho)'(z)\,\mathrm{d}z = 0,$$

by Cauchy–Goursat (Theorem 3.1.7), since U is simply connected by assumption, $\rho^{-1} \circ \psi^{-1} \circ \varphi^{-1}(\Gamma)$ is a closed piecewise C^1 curve in U, and the integrand is holomorphic on U. Therefore, $\tilde{U}$ is simply connected.

Hence, we may use our previous result and establish a biholomorphism $\tilde{f} : \tilde{U} \to \mathbb{D}$, unique up to a transformation in $\mathrm{Aut}(\mathbb{D})$. Let $f = \tilde{f} \circ \varphi \circ \psi \circ \rho$, which is a biholomorphism from U to $\mathbb{D}$. Similarly, it is unique up to a transformation in $\mathrm{Aut}(\mathbb{D})$, and the same assertion follows. $\square$

Remark: It is natural that we require $U \neq \mathbb{C}$; if there exists a univalent function $f : \mathbb{C} \to \mathbb{D}$, then by Liouville's Theorem (Theorem 3.2.3), f would be a constant function.

As we will see in @ sec:multivariatecomplexanalysis, this theorem and many other properties of one-variable holomorphic functions do not extend to functions of several complex variables.

5.4 The Schwarz–Christoffel Transformation

The Riemann Mapping Theorem is elegant in its own simplicity and definitions. However, it is only a theorem that guarantees existence of biholomorphisms. No information whatsoever can be straightforwardly extracted regarding the explicit construction of such biholomorphisms. However, in the explicit case that U is the open interior of a polygon, the result is provided by the Schwarz–Christoffel Transformation.

Let $a_1 < a_2 < \cdots < a_n$ be $n \in \mathbb{N}$ distinct real numbers. Suppose $\alpha_1, \alpha_2, ..., \alpha_n$ are n positive real numbers satisfying $\sum_{k=1}^{n} \alpha_k < n - 1$. Let

$$\beta(\zeta) = (\zeta - a_1)^{\alpha_1 - 1} \cdots (\zeta - a_n)^{\alpha_n - 1} = \prod_{k=1}^{n} (\zeta - a_k)^{\alpha_k - 1},$$

where the branch of each factor is selected to be

$$(\zeta - a_k)^{\alpha_k - 1} = \mathrm{e}^{(\alpha_k - 1)(\log(\zeta - a_k))},$$

where the branch of $\log(z)$ is selected such that $-\frac{\pi}{2} < \mathfrak{Im}(\log(z)) \leq 3\frac{\pi}{2}$, holomorphic on $\mathbb{C} \setminus \mathrm{i}\mathbb{R}_{\leq 0}$ (the lower imaginary axis is known as a *branch cut*). For $\zeta < a_k$, the argument of this factor is $\pi(\alpha_k - 1)$. For $\zeta < a_1$,

$$\arg(\beta(\zeta)) = \pi\left(-n + \sum_{k=1}^{n} \alpha_k\right),$$

achieved by selecting branches of each factor by the method described earlier.

Let k be fixed. If $\zeta \in (a_{k-1}, a_k)$, the branches of all $\left(\zeta - a_j\right)^{\alpha_j - 1}$ where $1 \leq j \leq k - 1$ have vanishing arguments; hence,

$$\arg(\beta(\zeta)) = \pi\left(-n + k - 1 + \sum_{j=k}^{n} \alpha_j\right).$$

If $\zeta > a_n$, we have

$$\arg(\beta(\zeta)) = 0.$$

Therefore, we can define $n + 2$ complex numbers via

$$w_0 = c\int_0^{-\infty} \beta(\zeta)\,\mathrm{d}\zeta, \quad w_k = c\int_0^{a_k} \beta(\zeta)\,\mathrm{d}\zeta, \quad w_{n+1} = c\int_0^{\infty} \beta(\zeta)\,\mathrm{d}\zeta$$

where $c \in \mathbb{R}_{>0}$ is fixed.

The absolute integrability of $\beta(\zeta)$ along the real axis concerns only the convergence at each singularity $\zeta = a_k$ and the behavior as $\zeta \to \pm\infty$. For a fixed k, $\beta(\zeta) = h_k(\zeta)(\zeta - a_k)^{\alpha_k - 1}$ (where h_k is holomorphic and nonzero in a compact neighborhood of a_k). Since $\alpha_k - 1 > -1$, it is an integrable singularity. Since $\beta(\zeta) \sim \zeta^{\sum \alpha_k - n}$ as $\zeta \to \pm\infty$ and $\sum_{k=1}^{n} \alpha_k - n < -1$, β is integrable on $\mathbb{R}$.

Let

$$f(z) = c \int^{z} \beta(\zeta) \, \mathrm{d}\zeta. \tag{5.4.1}$$

Since β is holomorphic on $\mathbb{H}^{+}$,

5.5 The Reflection Principle

We have previously considered analytic continuations over two regions with an intersection. Under certain conditions, analytic continuations can be derived across a curve, given by the following theorem.

Theorem 5.5.1 *(Painlevé)*: Let U_1 and U_2 be two disjoint simply connected open regions in $\mathbb{C}$ such that $\partial U_1 \cap \partial U_2$ is a simple curve γ without its endpoints. Let $f_1 : U_1 \to \mathbb{C}$ and $f_2 : U_2 \to \mathbb{C}$ be two holomorphic functions that are continuous on $U_1 \cup \gamma$ and $U_2 \cup \gamma$, respectively, such that $f_1 \equiv f_2$ on γ. Then there exists a unique holomorphic function

$$f = \begin{cases} f_1 & \text{on } U_1 \\ f_2 & \text{on } U_2 . \\ f_1 \equiv f_2 & \text{on } \gamma \end{cases}$$

on $U_1 \cup U_2 \cup \gamma$.

Proof: We aim to prove that the constructed function f is holomorphic on $U_1 \cup U_2 \cup \gamma$. In particular, we only need to prove that f is holomorphic on (a neighborhood of) γ, after which the Identity Theorem (Theorem 3.3.3) applies.

Let $z \in \gamma$ be fixed, and choose $R = R_z > 0$ such that $D(z, R) \subseteq U_1 \cup U_2 \cup \gamma$. Let Γ be any simple closed curve in $D(z, R)$. If Γ is fully contained in $U_1 \cup \gamma(\cap D(z, R))$, then by Cauchy–Goursat (Theorem 3.1.7),

$$\oint_{\Gamma} f(z) \, \mathrm{d}z = \oint_{\Gamma} f_1(z) \, \mathrm{d}z = 0.$$

Similarly, if Γ is fully contained in $U_2 \cup \gamma$, then

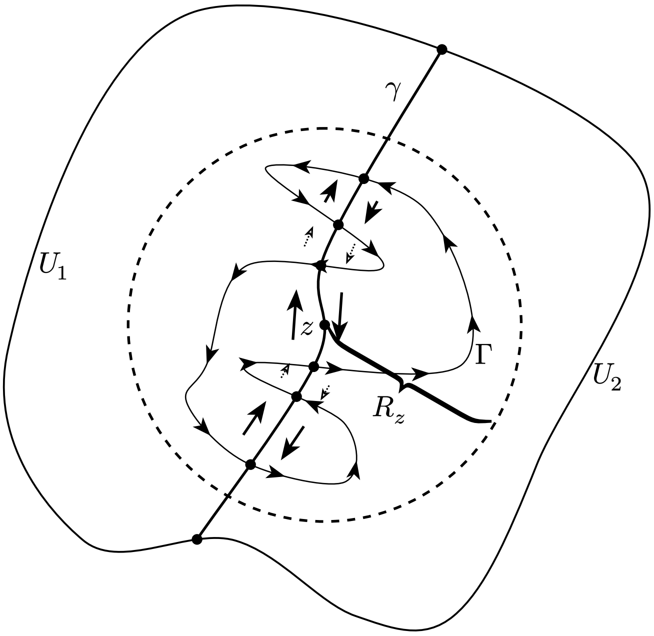

Figure 13: The two regions U_1 and U_2 sharing a boundary curve γ, the disk $D(z, R_z)$ for $z \in \gamma$, and the curve[1] Γ. Solid large arrows and arrowheads denote orientation of the $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ regions.

$$\oint_\Gamma f(z)\,\mathrm{d}z = \oint_\Gamma f_2(z)\,\mathrm{d}z = 0.$$

If Γ intersects γ, then we can decompose $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 is the part of Γ that lies in $U_1 \cup \gamma$ and Γ_2 is the part of Γ that lies in $U_2 \cup \gamma$. The set $\tilde{\Gamma} = \gamma \cap \operatorname{int}(\Gamma)$ closes Γ_1 and Γ_2 in the sense that $\tilde{\Gamma}_1 = \Gamma_1 \cup \tilde{\Gamma}$ and $\tilde{\Gamma}_2 = \Gamma_2 \cup \tilde{\Gamma}$ are both simple closed curves, or unions of simple closed curves (where $\tilde{\Gamma}$ in each of the two curves have opposite orientations, see Figure 13). By Cauchy–Goursat (Theorem 3.1.7), we have

$$\oint_\Gamma f(z)\,\mathrm{d}z = \left(\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\tilde{\Gamma}} - \int_{\tilde{\Gamma}}\right) f(z)\,\mathrm{d}z = \left(\oint_{\tilde{\Gamma}_1} + \oint_{\tilde{\Gamma}_2}\right) f(z)\,\mathrm{d}z = 0.$$

Hence, by Morera's Theorem (Theorem 3.2.4), f is holomorphic on $\bigcup_{z\in\gamma} D(z, R_z)$, and the assertion follows. ☐

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[1] Although more accurately, they are restricted to triangular paths. Our purpose here is to show that they intersect multiple times, the validity of the treatment remains the same.

A consequent result was discovered by Schwarz, known as the *reflection principle*, is a unique result derived from the above theorem for when the shared boundary curve lies in the real axis under certain conditions.

Theorem 5.5.2 *(Schwarz Reflection Principle)*: Let $U \subseteq \mathbb{C}$ be a connected region on one side of the real axis such that there exists a non-degenerate curve $\gamma \subseteq \partial U$ such that $\gamma \subseteq \mathbb{R}$. Let $f : U \to \mathbb{C}$ be holomorphic with continuity up to $U \cup \gamma$ such that f is real-valued on γ, and let $\tilde{U} = \{\overline{z} : z \in U\}$ be the reflection of U across the real axis. Then there exists a unique holomorphic function

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \in U \\ \overline{f(\overline{z})} & \text{if } z \in \tilde{U}. \\ f(z) \equiv \overline{f(\overline{z})} & \text{if } z \in \gamma \end{cases}$$

on $U \cup \tilde{U} \cup \gamma$.

<u>*Proof*</u>: If $z \in \mathbb{R}$, then $\overline{z} = z$, and since f is real on γ, it follows that $f(z) = \overline{f(\overline{z})}$ for $z \in \gamma$. Thus, we are left to prove that $z \mapsto \overline{f(\overline{z})}$ is holomorphic on $\tilde{U}$. Let $z_0 \in \tilde{U}$. It follows that

$$\lim_{\substack{z \to z_0 \\ z \in \tilde{U}}} \frac{\overline{f(\overline{z})} - \overline{f(\overline{z_0})}}{z - z_0} = \lim_{\substack{z \to z_0 \\ z \in \tilde{U}}} \overline{\left(\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}} \right)} = \overline{f'(\overline{z_0})}.$$

Since this limit exists, it follows that $\overline{f(\overline{z})}$ is holomorphic on $\tilde{U}$. Assume that $z_0 \in \gamma$. Since

$$\lim_{z \to z_0} \overline{f(\overline{z})} = \overline{f\left(\lim_{z \to z_0} \overline{z} \right)} = \overline{f(z_0)} = f(z_0),$$

it follows that $\overline{f(\overline{z})}$ is continuous on $\tilde{U} \cup \gamma$. Therefore, by the Painlevé Theorem, $\tilde{f}$ is holomorphic on $U \cup \tilde{U} \cup \gamma$. ☐

This conjugate-symmetry can be generalized by transforming γ:

Theorem 5.5.3 *(Symmetry Principle)*: Let $L \subset \mathbb{C}$ be an (infinite) straight line, and let $U \subset \mathbb{C}$ be an open region lying entirely on one side of L. Suppose $\gamma \subseteq L$ is a non-degenerate open curve contained in ∂U. If f is holomorphic on U, continuous on $U \cup \gamma$, and satisfies $f(\gamma) \subseteq \Gamma$, where $\Gamma \subset \mathbb{C}$ is a straight line, then there exists a unique holomorphic function $\tilde{f} : U \cup \tilde{U} \cup \gamma \to \mathbb{C}$ such that $\tilde{f} \equiv f$ on U, where $\tilde{U}$ is the reflection of U across L. Moreover, for any pair $z_1, z_2 \in U \cup \tilde{U} \cup \gamma$ symmetric with respect to L, the values $\tilde{f}(z_1)$ and $\tilde{f}(z_2)$ are symmetric with respect to Γ.

Proof: There exist $a, c \in \mathbb{C}^*$ and $b, d \in \mathbb{C}$ such that $\phi(z) = az + b$ maps L to $\mathbb{R}$ and $\psi(z) = cz + d$ maps Γ to $\mathbb{R}$. Let $U' = \phi(U)$, which lies entirely on one side of the real axis, and let $\gamma' = \phi(\gamma)$, a curve on the real axis. The function $\varphi = \psi \circ f \circ \phi^{-1}$ is holomorphic on U' and continuous on $U' \cup \gamma'$. By the Schwarz Reflection Principle (Theorem 5.5.2), there exists a unique holomorphic function $\tilde{\varphi} : U' \cup \widetilde{U'} \cup \gamma' \to \mathbb{C}$ such that $\tilde{\varphi} \equiv \varphi$ on U', where $\widetilde{U'}$ is the reflection of U' across the real axis. Then $\tilde{f} = \psi^{-1} \circ \tilde{\varphi} \circ \phi$ is a holomorphic function on $U \cup \tilde{U} \cup \gamma$ such that $\tilde{f} \equiv f$ on U. Since linear transformations preserves symmetry, for any pair $z_1, z_2 \in U \cup \tilde{U} \cup \gamma$ symmetric with respect to L, we have $\phi(z_1) = \overline{\phi(z_2)}$, and thus $\tilde{\varphi} \circ \phi(z_1)$ and $\tilde{\varphi} \circ \phi(z_2)$ are symmetric with respect to $\mathbb{R}$. Hence, $\tilde{f}(z_1)$ and $\tilde{f}(z_2)$ are symmetric with respect to $\psi^{-1}(\mathbb{R}) = \Gamma$. □

6 Rational Approximation Theory

By definition, a rational function is the quotient of two polynomials; and by Theorem 4.3.1, in equivalent formulation, it is a function meromorphic on all of $\hat{\mathbb{C}}$. The poles and zeros may not accumulate in $\hat{\mathbb{C}}$, and thus there are finitely many as a consequence of Bolzano–Weierstrass (Theorem 1.1.2).

When we refer to approximation, we refer to the approximation of a function as the uniform limit (of a sequence) of functions. Let $K \subseteq \mathbb{C}$ be compact and suppose $f : K \to \mathbb{C}$ is a given function on K. As a consequence of Mergelyan's Theorem (Theorem 6.2.2), sufficient conditions for f to be the uniform limit of rational functions whose poles lie in (a subset of given points of) $\hat{\mathbb{C}} \setminus K$ are the continuity of f on K and the holomorphy of f on $\overset{\circ}{K}$.

6.1 Runge's Theorem

In the earliest formulation by Carl Runge in 1885, he provided the sufficiency of the holomorphy of f on K (in effect, a neighborhood of K).

The proof can be well-organized through the use of the results that we will now introduce. In essence, it involves applying Cauchy–Goursat to f and the subsequent use of Riemann sums to approximate the complex line integral.

Proposition 6.1.1: Let $K \subseteq \mathbb{C}$ be compact and suppose $U \supset K$ is a neighborhood of K that is relatively compact in $\mathbb{C}$. Let $f : U \to \mathbb{C}$ be an arbitrary holomorphic function. Then for fixed $\varepsilon > 0$, there exists a rational function $\psi(z)$ with only simple poles (all of which lie in $\mathbb{C} \setminus K$) such that

$$\lim_{z \to \infty} \psi(z) = 0, \quad \sup_{z \in K} |f(z) - \psi(z)| < \varepsilon.$$

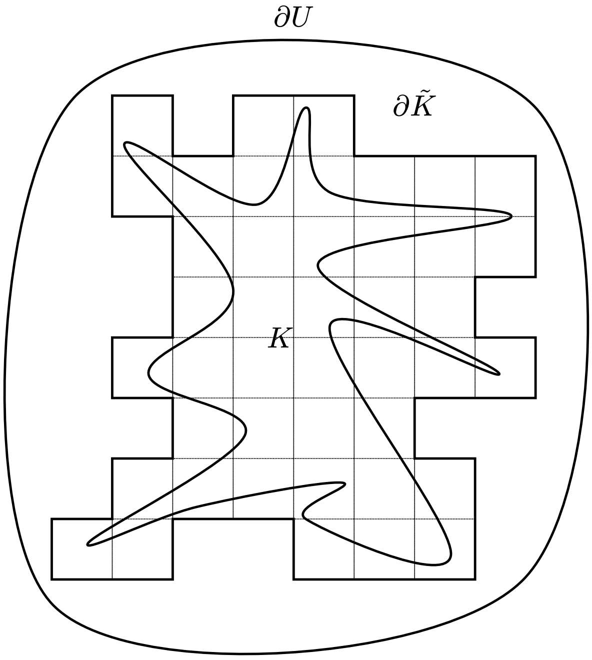

Figure 14: The elements of $\mathcal{G}$, relative to K and its neighborhood, U.

Proof: By assumption of relative compactness, $\sigma = \operatorname{dist}(\partial U, \partial K)$, or the distance (infimum) between K and $\mathbb{C} \setminus U$, is positive and finite. More concretely, let

$$\sigma = \inf\{|z_1 - z_2| : z_1 \in K, z_2 \in \mathbb{C} \setminus U\} > 0.$$

The longest distance between two points in any square is the length of the diagonal. Hence, any square Q that intersects ∂K with a side length less than $\sigma \frac{\sqrt{2}}{2}$ will lie completely within U.

Choose $m \in \mathbb{N}$ to satisfy $2^{1-m} < \sigma$ and consider the grid generated by compact squares in the form of

$$\left\{ x + \mathrm{i}y : \frac{j}{2^m} \le x \le \frac{j+1}{2^m}, \frac{k}{2^m} \le y \le \frac{k+1}{2^m} \right\}$$

(where j and k are integers) and let $\mathcal{G}$ be the collection of all such squares in this grid that intersect K, and it follows that $\tilde{K} = \bigcup_{Q \in \mathcal{G}} Q \subset U$ (refer to Figure 14).

As a consequence of Cauchy–Goursat (Theorem 3.1.8), we have

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial \tilde{K}} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} = f(z) \tag{6.1.1}$$

in the case that $z \in \tilde{K}$. The boundary $\partial \tilde{K}$ may be written as the union of n lines parameterized by $0 \leq t \leq 1$; more concretely, we have $\partial \tilde{K} = \bigcup_{j \in \mathbb{N}_{\leq n}} \gamma_j([0,1])$. Hence we have in equivalent formulation,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{n} \int_{\gamma_j([0,1])} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{n} \int_0^1 \frac{f\big(\gamma_j(t)\big)\gamma_j'(t)}{\gamma_j(t) - z}\,\mathrm{d}t.$$

The distance between K and $\partial \tilde{K}$ is strictly positive. Suppose instead that the distance were zero. Then some point of K would lie on the boundary of a square $Q \in \mathcal{G}$ that intersects $\partial \tilde{K}$. If this point lies on an edge of Q (but not at a vertex), then the square adjacent along that edge must also intersect K, and hence belong to $\mathcal{G}$, contradicting the assumption that the point lies on $\partial \tilde{K}$. If the point lies at a vertex of Q, then all three adjacent squares also intersect K, so they too belong to $\mathcal{G}$, leading to the same contradiction. Thus, the distance must be positive.

Hence, each integrand as defined in (6.1.1) is jointly continuous for $t \in [0,1]$ and $z \in K$. By compactness of the product, it is in fact uniformly continuous by Heine–Cantor (Theorem 1.2.15).

Hence, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall z \in K$, $\forall 1 \leq j \leq n$ (uniform in j as we can take the minimum of each δ_j), and $\forall t_1, t_2 \in [0,1]$ satisfying $|t_1 - t_2| < \delta$,

$$\left| \frac{f\big(\gamma_j(t_1)\big)\gamma_j'(t_1)}{\gamma_j(t_1) - z} - \frac{f\big(\gamma_j(t_2)\big)\gamma_j'(t_2)}{\gamma_j(t_2) - z} \right| < \frac{\varepsilon}{n}.$$

Partition $[0,1]$ by $0 = t_0 < t_1 < \ldots < t_m = 1$ such that $\forall 0 \leq k < m$, $\Delta t_k = t_{k+1} - t_k < \delta$. It follows that

$$\begin{aligned}
&\left| f(z) - \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{n} \sum_{k=0}^{m-1} \frac{f\big(\gamma_j(t_k)\big)\gamma_j'(t_k)}{\gamma_j(t_k) - z} \Delta t_k \right| \\
&\quad = \frac{1}{2\pi} \left| \sum_{j=1}^{n} \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \left[\frac{f\big(\gamma_j(t)\big)\gamma_j'(t)}{\gamma_j(t) - z} - \frac{f\big(\gamma_j(t_k)\big)\gamma_j'(t_k)}{\gamma_j(t_k) - z} \right] \mathrm{d}t \right| \\
&\quad \leq \frac{1}{2\pi} \sum_{j=1}^{n} \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \left| \frac{f\big(\gamma_j(t)\big)\gamma_j'(t)}{\gamma_j(t) - z} - \frac{f\big(\gamma_j(t_k)\big)\gamma_j'(t_k)}{\gamma_j(t_k) - z} \right| \mathrm{d}t \\
&\quad \leq \frac{\varepsilon}{2n\pi} \sum_{j=1}^{n} \sum_{k=0}^{m-1} \Delta t_k = \frac{\varepsilon}{2\pi} < \varepsilon
\end{aligned}$$

uniformly in $z \in K$. The summation

$$\psi(z) = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{n} \sum_{k=0}^{m-1} \frac{f(\gamma_j(t_k))\gamma_j'(t_k)}{\gamma_j(t_k) - z} \Delta t_k$$

defines a rational function with simple poles at each $\gamma_j(t_k) \in \partial \tilde{K}$, which is disjoint from K. □

In its full generality, we will now apply a technique to push a pole to a prescribed point, while ensuring that the resulting function remains uniformly approximated outside of a given connected compact set that contains both the original and target pole locations.

Lemma 6.1.1 *(Pole-Pushing Lemma)*: Let $\alpha, \beta \in \mathbb{C}$ and let $f(z)$ be a rational function with a single singularity, a pole at $z = \alpha$, whose Laurent expansion consists solely of its principal part. Then $\forall r > |\alpha - \beta|$, $\forall \varepsilon > 0$, there exists a rational function $\psi(z)$ whose only singularity is a pole at $z = \beta$ such that

$$\sup_{z \in \hat{\mathbb{C}} \setminus D(\beta, r)} |f(z) - \psi(z)| < \varepsilon.$$

Proof: By assumption, f can be expressed as a polynomial of

$$\begin{aligned} (z-\alpha)^{-1} &= (z-\beta)^{-1} \frac{1}{(z-\alpha)(z-\beta)^{-1}} \\ &= (z-\beta)^{-1} \frac{1}{1 - (\alpha-\beta)(z-\beta)^{-1}} \\ &= (z-\beta)^{-1} \sum_{k=0}^{\infty} \left(\frac{\alpha - \beta}{z - \beta} \right)^k. \end{aligned}$$

This series locally uniformly converges on

$$\left\{ z \in \hat{\mathbb{C}} : |\alpha - \beta||z - \beta|^{-1} < 1 \right\} = \hat{\mathbb{C}} \setminus \overline{D(\beta, |\alpha - \beta|)}$$

and uniformly converges on $\hat{\mathbb{C}} \setminus D(\beta, r)$. Hence, for $m \in \mathbb{N}$, we have

$$f(z) = \sum_{j=1}^{m} a_{-j} \left((z-\beta)^{-1} \sum_{k=0}^{\infty} \left(\frac{\alpha - \beta}{z - \beta} \right)^k \right)^j.$$

For fixed j (where $a_{-j} \neq 0$), we aim to prove the existence of an $N \in \mathbb{N}$ such that $\forall n > N$, we have at least

$$\begin{aligned}\Bigg|&\left[(z-\beta)^{-1}\sum_{k=0}^{\infty}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right]^{j}\\&\qquad-\left[(z-\beta)^{-1}\sum_{k=0}^{n}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right]^{j}\Bigg|<\frac{\varepsilon}{m\left|a_{-j}\right|},\end{aligned}\tag{6.1.2}$$

where $z\in\hat{\mathbb{C}}\setminus D(\beta,r)$. Since $\left|\frac{1}{z-\beta}\right|<\frac{1}{r}$, we can restrict (6.1.2) further with

$$\left|\left(\sum_{k=0}^{\infty}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right)^{j}-\left(\sum_{k=0}^{n}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right)^{j}\right|<r^{j}\frac{\varepsilon}{m\left|a_{-j}\right|}.$$

Additionally, the difference on the left-hand side is also equal to

$$\begin{aligned}&\left|\sum_{k=n+1}^{\infty}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right|\\&\qquad\cdot\left|\sum_{l=0}^{j-1}\left(\sum_{k=0}^{n}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right)^{l}\left(\sum_{k=0}^{\infty}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right)^{j-l-1}\right|.\end{aligned}\tag{6.1.3}$$

For any $n\in\mathbb{N}$, we have

$$\left|\sum_{k=0}^{n}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right|\leq\sum_{k=0}^{n}\left|\frac{\alpha-\beta}{z-\beta}\right|^{k}\leq\sum_{k=0}^{\infty}\left|\alpha-\frac{\beta}{r}\right|^{k}\leq\frac{1}{1-\left|\alpha-\frac{\beta}{r}\right|}.$$

Since the dominating sequence of partial sums are monotonically increasing, it follows that the sequence of partial sums is uniformly bounded by

$$M=\frac{r}{r-|\alpha-\beta|}$$

on $\hat{\mathbb{C}}\setminus D(\beta,r)$. Thus, (6.1.3) is bounded by $M^{j-1}j\left|\sum_{k=n+1}^{\infty}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right|$, and we apply further restriction by setting this to be bounded by $r^{j}\frac{\varepsilon}{m\left|a_{-j}\right|}$. By uniform convergence, for any $\varepsilon>0$, $\exists N_{j}\in\mathbb{N}$ such that $\forall n>N_{j}$,

$$\left|\sum_{k=n+1}^{\infty}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right|<r^{j}\frac{\varepsilon}{M^{j-1}jm\left|a_{-j}\right|}.$$

For $n>N_{j}$, (6.1.2) is satisfied, and $\forall n>\max_{\substack{j\in\mathbb{N}_{\leq m}\\ a_{-j}\neq 0}}\left(N_{j}\right)$, $z\in\hat{\mathbb{C}}\setminus D(\beta,r)$, we have

$$\begin{aligned}
&\left|f(z)-\sum_{j=1}^{m} a_{-j}\left(\left(\frac{1}{z-\beta}\sum_{k=0}^{n}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right)^{j}\right)\right| \\
&\qquad \leq \sum_{\substack{j=1 \\ a_{-j}\neq 0}}^{m} \left|a_{-j}\right| \left|\left(\sum_{k=0}^{\infty}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right)^{j}-\left(\sum_{k=0}^{n}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right)^{j}\right| \\
&\qquad \leq \sum_{\substack{j=1 \\ a_{-j}\neq 0}}^{m} \left|a_{-j}\right| \frac{\varepsilon}{m\left|a_{-j}\right|} \leq \varepsilon,
\end{aligned}$$

which completes the proof as

$$\psi(z)=\sum_{j=1}^{m} a_{-j}\left((z-\beta)^{-1}\sum_{k=0}^{n}\left(\frac{\alpha-\beta}{z-\beta}\right)^{k}\right)^{j}$$

is rational with a pole at $z=\beta$. □

Lemma 6.1.2 *(Generalized Pole-Pushing Lemma)*: Let $K \subseteq \mathbb{C}$ be compact and choose $a \in \mathbb{C} \setminus K$. Let U be the connected component of $\hat{\mathbb{C}} \setminus K$ containing a. Then $\forall \varepsilon > 0$, $\forall \zeta \in U$, there exists a rational function ψ with a pole only at ζ such that

$$\sup_{z\in K}\left|\frac{1}{z-a}-\psi(z)\right| < \varepsilon.$$

Proof: Define the set

$$S=\left\{\zeta \in U\setminus\{\infty\} : (\forall\varepsilon>0)(\exists\psi)\begin{bmatrix}\psi \text{ is rational,} \\ \psi(\mathbb{C}\setminus\{\zeta\})\subseteq\mathbb{C}\wedge\psi(\zeta)=\infty \\ \sup_{z\in K}\left|\frac{1}{z-a}-\psi(z)\right|<\varepsilon\end{bmatrix}\right\}.$$

Since $a \in U$ satisfies the condition with $\psi(z)=\frac{1}{z-a}$, it follows that $a \in S$, ensuring that S is nonempty.

Consider $\zeta \in S$, where ζ lies in the complement of K. The distance from ζ to K, denoted $\operatorname{dist}(\zeta, K)$, is positive, and the open disk $D(\zeta, \operatorname{dist}(\zeta, K))$ is disjoint from K. Let ζ' be an arbitrary point in this disk. By the definition of S, for every $\varepsilon > 0$, there exists a rational function ψ with a pole only at ζ such that

$$\sup_{z\in K}\left|\frac{1}{z-a}-\psi(z)\right| < \frac{\varepsilon}{2}.$$

By Lemma 6.1.1, there exists a rational function ϕ with a pole only at ζ' such that

$$\sup_{z \in \hat{\mathbb{C}} \setminus D(\zeta, \operatorname{dist}(\zeta, K))} |\phi(z) - \psi(z)| < \frac{\varepsilon}{2},$$

which implies

$$\sup_{z \in K} |\phi(z) - \psi(z)| < \frac{\varepsilon}{2}.$$

Thus,

$$\sup_{z \in K} \left| \frac{1}{z-a} - \phi(z) \right| \leq \sup_{z \in K} \left| \frac{1}{z-a} - \psi(z) \right| + \sup_{z \in K} |\psi(z) - \phi(z)| < \varepsilon,$$

and by definition, $\zeta' \in S \implies D(\zeta, \operatorname{dist}(\zeta, K)) \subseteq S$. Hence, S is relatively open in $U \setminus \{\infty\}$.

Now, consider $\zeta \in U \setminus (S \cup \{\infty\})$. Suppose there exists $\zeta' \in D(\zeta, \operatorname{dist}(\zeta, K)) \cap S$. By repeated application of the preceding argument, this would imply $\zeta \in S$, contradicting the assumption that $\zeta \in U \setminus (S \cup \{\infty\})$. Therefore, no such ζ' exists, and S is relatively closed in $U \setminus \{\infty\}$.

Since $U \setminus \{\infty\}$ is connected and S is both relatively open and closed in $U \setminus \{\infty\}$, it follows from Theorem 3.2.1.3 that $S = U \setminus \{\infty\}$, completing the proof under the assumption that $\infty \notin U$.

Now suppose $\infty \in U$. In essence, we pole push to a point outside a disk on which we can make approximations by Taylor polynomials. Let $R > 0$ satisfy $K \subset D(0, R)$ and let $b \in U \setminus \left(\{\infty\} \cup \overline{D(0, R)}\right)$ be an arbitrary point. By Lemma 6.1.1, there exists some rational function $\tilde{\psi}(z)$ with a pole at b such that

$$\sup_{z \in K} \left| \tilde{\psi}(z) - \frac{1}{z-a} \right| < \frac{\varepsilon}{2}.$$

Since $\tilde{\psi}$ is holomorphic on some neighborhood of $\overline{D(0, R)}$, we have

$$\tilde{\psi}(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{on} \quad \overline{D(0, R)},$$

and it uniformly converges on $\overline{D(0, R)}$. Hence, $\exists N \in \mathbb{N}$ such that

$$\sup_{z \in \overline{D(0,R)}} \left| \tilde{\psi}(z) - \sum_{k=0}^{N} a_k z^k \right| < \frac{\varepsilon}{2}.$$

Since polynomials have poles at ∞, we have

$$\begin{aligned}\sup_{z\in\overline{D(0,R)}}\left|\frac{1}{z-a}-\sum_{k=0}^{N}a_kz^k\right| &\leq \sup_{z\in\overline{D(0,R)}}\left|\frac{1}{z-a}-\tilde{\psi}(z)\right| \\ &\quad+\sup_{z\in\overline{D(0,R)}}\left|\tilde{\psi}(z)-\sum_{k=0}^{N}a_kz^k\right|<\varepsilon.\end{aligned}$$

☐

Theorem 6.1.1 *(Runge)*: Let $K\subseteq\mathbb{C}$ be compact such that $\hat{\mathbb{C}}\setminus K$ has finitely many connected components and suppose f is holomorphic on a neighborhood of K. Let E be a subset of $\hat{\mathbb{C}}\setminus K$ containing one point from each of its connected components. Then $\forall\varepsilon>0$, there is a rational function ψ whose poles lie in E such that

$$\sup_{z\in K}|f(z)-\psi(z)|<\varepsilon.$$

Proof: By Proposition 6.1.1, there is a rational function ϕ with simple poles in $\mathbb{C}\setminus K$ satisfying $\phi(\infty)=0$ such that

$$\sup_{z\in K}|f(z)-\phi(z)|<\frac{\varepsilon}{2}. \tag{6.1.4}$$

Let the poles of ϕ be $\{\beta_k\}_{k\in\mathbb{N}_{\leq n}}\subseteq\mathbb{C}\setminus K$, and as a consequence, we have $\phi(z)=\sum_{k=1}^{n}\frac{a_k}{z-\beta_k}+\varphi(z)$ where $\varphi(z)$ is entire. Since $\phi(\infty)=0$, we have $\varphi\equiv0$ by Liouville's Theorem (Theorem 3.2.3). By Lemma 6.1.2, there exist rational functions $\{\psi_k\}_{k\in\mathbb{N}_{\leq n}}$ whose only poles lie in E such that $\forall k$,

$$\sup_{z\in K}\left|\frac{1}{z-\beta_k}-\psi_k(z)\right|<\frac{\varepsilon}{2n|a_k|}$$

and it follows that

$$\sup_{z\in K}\left|\phi(z)-\sum_{k=1}^{n}a_k\psi_k(z)\right|\leq\sup_{z\in K}\sum_{k=1}^{n}\left|\frac{a_k}{z-\beta_k}-a_k\psi_k(z)\right|<\frac{\varepsilon}{2}.$$

Let $\psi(z)=\sum_{k=1}^{n}a_k\psi_k(z)$. From (6.1.4), we have

$$\sup_{z\in K}|f(z)-\psi(z)|\leq\sup_{z\in K}|f(z)-\phi(z)|+\sup_{z\in K}|\phi(z)-\psi(z)|<\varepsilon.$$

☐

6.2 Mergelyan's Theorem

Although many mathematicians have since tried after the efforts of Weierstrass and Runge to approximation continuous functions holomorphic on the interior restriction, it was only 67 years later when Armenian mathematician provided the first widely accepted proof. The proof of Runge's Theorem (specifically in Proposition 6.1.1) relied heavily on the assumption of holomorphy on a neighborhood, a rational function was created by placing poles in prescribed points of a contour that lay outside of K but within its domain of holomorphy. Obviously, these assumptions are null under the context of this new formulation.

The proof proposed by Mergelyan is almost trivial when compared with the results of many other mathematicians at the time. It even uses the concepts previously proposed by Runge. This begs the question: why was there such a prolonged time gap between the two similar formulations? Many mathematicians felt that the conclusion was "too good to be true"; during this elapsed time period there were many efforts of mathematicians that resulted in many technical partial results. Mergelyan's Theorem came as a surprise as it encapsulated many of those results with simplicity.

As we have previously seen, there is a prevalent notion in complex analysis that regards ∞ intrinsically as essentially any other point of $\hat{\mathbb{C}}$. An appertaining question relates to the complex derivative at ∞. Although

$$f'(\infty) \overset{?}{=} \lim_{z \to \infty} f'(z)$$

may seem to be a natural object to consider, it is quite impractical; there exist functions which decay quickly to 0, while $f'(z)$ is unbounded as $z \to \infty$ (take $z \mapsto \frac{\sin(z^2)}{\sqrt{z}}$ as an example). Even the assumption that $\lim_{z\to\infty} f'(z) = 0$ does not imply that $f(z)$ has a removable singularity at ∞ (consider $z \mapsto \sqrt{z}$).

Definition 6.2.1: Let $R > 0$, $f : \mathbb{C} \setminus \overline{D(0, R)} \to \mathbb{C}$ be holomorphic such that f has a removable singularity at ∞. Then we define the derivatives of f at ∞ to be

$$f^{(n)}(\infty) = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(f\left(\frac{1}{z}\right)\right)\Bigg|_{z=0}.$$

In the case that $n = 1$, we have

$$f'(\infty) = -\lim_{z \to \infty} z^2 f'(z). \tag{6.2.1}$$

This is precisely the first singular term of the Laurent expansion of f at ∞.

Remark: This definition may feel unsatisfactory, but the underlying logic here is similar to the method used to generalize residues to ∞.

If f is bijective and meromorphic on some neighborhood of a point $a \in \mathbb{C}$ such that $f(a) = \infty$, then we informally define the derivative at the pole a to be

$$f'(a) = \frac{1}{\left(f^{-1}\right)'(\infty)} = -\lim_{w \to \infty} \frac{1}{w^2 \left(f^{-1}\right)'(w)} \\ = -\lim_{w \to \infty} \frac{f'\left(f^{-1}(w)\right)}{w^2}. \tag{6.2.2}$$

Let $z = \left(f^{-1}\right)(w)$. Then we have

$$f'(a) = -\lim_{z \to a} \frac{f'(z)}{f(z)^2} = \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{1}{f(z)} \right) \Bigg|_{z=a}. \tag{6.2.3}$$

Proposition 6.2.1: For any connected compact set $K \subseteq \mathbb{C}$ containing at least two distinct points such that $\hat{\mathbb{C}} \setminus K$ is connected, let ϕ be an arbitrary biholomorphism mapping $\hat{\mathbb{C}} \setminus K$ to $\mathbb{D}$ such that $\phi(\infty) = \lim_{z \to \infty} \phi(z) = 0$. It follows that

$$|\phi'(\infty)| \geq \frac{1}{4} \operatorname{diam}(K),$$

where $\operatorname{diam}(K) = \sup_{z, \zeta \in K} |\zeta - z|$.

Proof: Denote the derivative of ϕ at the infinity to be α. By (6.2.2), we have

$$\left(\phi^{-1}\right)'(0) = \frac{1}{\phi'(\infty)} = \frac{1}{\alpha} = -\lim_{z \to 0} \frac{\left(\phi^{-1}\right)'(z)}{\phi^{-1}(z)^2} \Longleftrightarrow -\lim_{z \to 0} \frac{\phi^{-1}(z)^2}{\alpha \left(\phi^{-1}\right)'(z)} = 1.$$

Fix $\tau \in K$ and let $\psi(z) = \frac{\alpha}{\phi^{-1}(z) - \tau}$, univalent on $\mathbb{D}$. By direct calculation, we have $\psi(0) = 0$. Additionally,

$$\psi'(0) = -\lim_{z \to 0} \frac{\alpha \left(\phi^{-1}\right)'(z)}{\left(\phi^{-1}(z) - \tau\right)^2} = \lim_{z \to 0} \frac{\alpha \left(\phi^{-1}\right)'(z)}{\left(\phi^{-1}(z) - \tau\right)^2} \cdot \frac{\phi^{-1}(z)^2}{\alpha \left(\phi^{-1}\right)'(z)} = 1.$$

By the Koebe Quarter Theorem (@ thm:koebequarter), whose proof is independent of results of this section, in accordance, $D\left(0, \frac{1}{4}\right) \subseteq \psi(\mathbb{D})$. Let $\mu \in K \setminus \{\tau\}$. Obviously, $\mu \notin \left(\phi^{-1}\right)(\mathbb{D}) = \hat{\mathbb{C}} \setminus K$.

Let $z \mapsto \frac{\alpha}{z - \tau}$ be injective on $\hat{\mathbb{C}}$. For the sake of contradiction, assume that $\left(z \mapsto \frac{\alpha}{z - \tau}\right)(\mu) \in \psi(\mathbb{D})$. Then $\exists \zeta \in \left(\phi^{-1}\right)(\mathbb{D})$ such that $\frac{\alpha}{\zeta - \tau} = \frac{\alpha}{\mu - \tau}$. By injectivity, $\zeta = \mu$, which contradicts $\mu \in K$, and accordingly, $\frac{\alpha}{\mu - \tau} \notin \psi(\mathbb{D}) \supseteq D\left(0, \frac{1}{4}\right)$.

Hence,

$$\left| \frac{\alpha}{\mu - \tau} \right| \geq \frac{1}{4} \Longleftrightarrow |\alpha| \geq \frac{|\mu - \tau|}{4}.$$

By taking the supremum for $\mu, \tau \in K$, the proof is complete. □

Remark: Such a biholomorphism will always exist; for arbitrary $\zeta \in K$, the map $z \mapsto \frac{1}{z-\zeta}$ maps $\hat{\mathbb{C}} \setminus K$ to a simply connected, proper subset of $\mathbb{C}$, which is biholomorphic to $\mathbb{D}$ by the Riemann Mapping Theorem (Theorem 5.3.1).

Proposition 6.2.2: Let $a \in \mathbb{C}$, $r > 0$, and suppose $K \subseteq D(a, r)$ is compact such that $\hat{\mathbb{C}} \setminus K$ is connected and $\operatorname{diam}(K) \geq \frac{r}{2}$. Then there is a family of holomorphic functions $\mathcal{F} = \left\{\varphi_\zeta\right\}_{\zeta \in D(a,r)}$, where $\forall \zeta \in D(a, r)$,

$$\varphi_\zeta : \hat{\mathbb{C}} \setminus K \to \mathbb{C},$$

and

1 $\left|\varphi_\zeta(z)\right| \leq \frac{584}{r}$ for any $z \in \hat{\mathbb{C}} \setminus K$.
2 $\left|\varphi_\zeta(z) - \frac{1}{z-\zeta}\right| \leq \frac{4676 r^2}{|\zeta - z|^3}$ for any $z \in \hat{\mathbb{C}} \setminus (K \cup \{\zeta\})$.
3 The function $\varphi(\zeta, z) \equiv \varphi_\zeta(z)$ is jointly continuous in ζ and z.

Proof: For brevity, assume $a = 0$.

Let $\tilde{\varphi}$ be a conformal mapping from $\hat{\mathbb{C}} \setminus K$ to $\mathbb{D}$, such that $\tilde{\varphi}(\infty) = 0$ and $\alpha = \tilde{\varphi}'(\infty) \in \mathbb{R}_{>0}$. Let $\varphi(z) = \frac{1}{\alpha}\tilde{\varphi}(z)$. It follows that $\varphi'(\infty) = 1$, $\varphi(\infty) = 0$. By Proposition 6.2.1,

$$|\alpha| \geq \frac{1}{4} \operatorname{diam}(K) \Longleftrightarrow |\varphi(z)| \leq \frac{4|\tilde{\varphi}(z)|}{\operatorname{diam}(K)}.$$

Consequently, we have the crucial estimate of $\varphi\left(\hat{\mathbb{C}} \setminus K\right) \subseteq D\left(0, \frac{4}{\operatorname{diam}(K)}\right) \subseteq D\left(0, \frac{8}{r}\right)$. For fixed $\zeta \in D(0, r)$, define

$$\varphi_\zeta(z) = \varphi(z) + (\zeta - \beta)\varphi^2(z), \quad z \in \hat{\mathbb{C}} \setminus K$$

where $\beta = \frac{\varphi''(\infty)}{2}$. The application of Cauchy's Estimate (Theorem 3.2.2) on $\left(z \mapsto \frac{1}{z}\right)\left(\hat{\mathbb{C}} \setminus D(0, r)\right) = D\left(0, \frac{1}{r}\right)$ gives:

$$|\beta| = \frac{1}{2}\left|\frac{\mathrm{d}^2}{\mathrm{d}z^2}\varphi\left(\frac{1}{z}\right)\bigg|_{z=0}\right| \leq \frac{\sup_{D(0,\frac{1}{r})}\left|\varphi\left(\frac{1}{z}\right)\right|}{\operatorname{dist}\left(0, \partial D\left(0, \frac{1}{r}\right)\right)^2} = 8r.$$

Hence,

$$\begin{aligned} \left|\varphi_\zeta(z)\right| &\leq |\varphi(z)| + |\zeta - \beta||\varphi(z)|^2 \leq |\varphi(z)| + |\zeta - \beta||\varphi(z)|^2 \\ &\leq \frac{8}{r} + 9r\frac{64}{r^2} = \frac{584}{r}. \end{aligned}$$

This is Part 1. Suppose that $|z - \zeta| > 2r$. It follows from $|\zeta| < r$ that $|z| > r$ (from the reverse triangle inequality) and hence disjoint from K and ζ. On this infinite annulus, we have the Laurent expansion (from Theorem 4.1.2) that

$$\varphi(z) = \sum_{k=1}^{\infty} \frac{\mu_k}{(z-\zeta)^k} = \frac{1}{z-\zeta} + \frac{\mu}{(z-\zeta)^2} + \mathcal{O}\left(\frac{1}{(z-\zeta)^3}\right)$$

where $\mu_1 = 1$ because $\varphi \sim \frac{1}{z}$ as $z \to \infty$. Since $|z| > r$, we have the global Laurent expansion

$$\varphi(z) = \frac{1}{z} + \frac{\beta}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right).$$

Hence,

$$\begin{aligned}
\frac{1}{z-\zeta} + \frac{\mu}{(z-\zeta)^2} &= \frac{1}{z} + \frac{\beta}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \\
z + \zeta + \mu &= z + \frac{\zeta^2}{z} + \beta + \frac{\beta\zeta^2}{z^2} - \frac{2\beta\zeta}{z} + \mathcal{O}\left(\frac{1}{z}\right) = z + \beta + \mathcal{O}\left(\frac{1}{z}\right) \\
\mu &= \beta - \zeta
\end{aligned}$$

by letting $z \to \infty$. Since

$$\varphi(z)^2 = \left(\frac{1}{z-\zeta} + \mathcal{O}\left(\frac{1}{(z-\zeta)^2}\right)\right)^2 = \frac{1}{(z-\zeta)^2} + \mathcal{O}\left(\frac{1}{(z-\zeta)^3}\right),$$

from the definition of φ_ζ, we have

$$\begin{aligned}
\varphi_\zeta(z) - \frac{1}{z-\zeta} = \varphi - \mu\varphi^2 - \frac{1}{z-\zeta} &= \frac{\mu}{(z-\zeta)^2} + \mathcal{O}\left(\frac{1}{(z-\zeta)^3}\right) \\
&\quad - \frac{\mu}{(z-\zeta)^2} - \mathcal{O}\left(\frac{\mu}{(z-\zeta)^3}\right) \\
&= \mathcal{O}\left(\frac{1}{(z-\zeta)^3}\right).
\end{aligned}$$

Hence, there exists some $M > 0$ such that

$$\left|\varphi_\zeta(z) - \frac{1}{z-\zeta}\right| < \frac{M}{|z-\zeta|^3} \iff \left|\varphi_\zeta(z) - \frac{1}{z-\zeta}\right| |z-\zeta|^3 < M$$

for all z satisfying $|z-\zeta| > 2r$. By Theorem 3.2.6, $\left(\varphi_\zeta(z) - \frac{1}{z-\zeta}\right)(z-\zeta)^3$ has a removable singularity at ∞. On the other hand, for $|z-\zeta| \leq 2r$ such that $z \in \hat{\mathbb{C}} \setminus (K \cup \{\zeta\})$, we have

$$\begin{aligned}
\left|\varphi_\zeta(z) - \frac{1}{z-\zeta}\right| |z-\zeta|^3 &\leq |\varphi_\zeta(z)||z-\zeta|^3 + |z-\zeta|^2 \\
&\leq \frac{584}{r}(2r)^3 + (2r)^2 = 4676r^2
\end{aligned}$$

from Part 1. The Maximum Modulus Principle (Theorem 3.4.1) implies that

$$\sup_{|z-\zeta|>2r}\left|\varphi_\zeta(z)-\frac{1}{z-\zeta}\right||z-\zeta|^3 \le \sup_{|z-\zeta|=2r}\left|\varphi_\zeta(z)-\frac{1}{z-\zeta}\right||z-\zeta|^3 \le 4676r^2$$

and thus Part 2 follows. The joint continuity of φ_ζ is immediate from the definition.

Lastly, if $a \neq 0$, we may define $\varphi_\zeta(z) = \tilde{\varphi}_{\zeta-a}(z-a)$ where $\{\tilde{\varphi}_{\zeta-a}\}$ is the family constructed above for the set $\{z-a : z \in K\} \subset D(0,r)$. ☐

Proposition 6.2.3: Suppose

$$\lambda(z) = \begin{cases} (1-|z|^2)^2 |z| < 1, \\ 0 \qquad\qquad |z| \ge 1, \end{cases} \quad \lambda_r(z) = \frac{3}{\pi r^2}\lambda\left(\frac{z}{r}\right) \quad \forall r > 0 \tag{6.2.4}$$

For fixed r, the function λ_r satisfies:

1 $\iint_{\mathbb{C}} \lambda_r(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta = 1$, where $\zeta = \xi + \mathrm{i}\eta$.
2 $\lambda_r \in C^1(\mathbb{C})$ and is compactly supported.
3 $\iint_{\mathbb{C}} \frac{\partial\lambda_r}{\partial\bar{\zeta}}\,\mathrm{d}\xi\,\mathrm{d}\eta = 0$.
4 $\iint_{\mathbb{C}} \left|\frac{\partial\lambda_r}{\partial\bar{\zeta}}\right|\mathrm{d}\xi\,\mathrm{d}\eta \le \frac{2\pi}{r}$.
5 $\|\nabla\lambda_r(z)\| \le \frac{4}{r^3}$ for all z, where $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ denotes the vector differential operator.
6 For any $z \in \mathbb{C}$ such that f is a holomorphic function on $D(z,r)$, we have the integral formula.

$$f(z) = \iint_{D(0,r)} f(z-\zeta)\lambda_r(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{6.2.5}$$

Proof: Let $\zeta = \rho\mathrm{e}^{\mathrm{i}\theta}$. Then we have

$$\begin{aligned} \iint_{\mathbb{C}} \lambda_r(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta &= \int_0^{2\pi}\int_0^r \lambda_r(\rho\mathrm{e}^{\mathrm{i}\theta})\rho\,\mathrm{d}r d\theta \\ &= \int_0^{2\pi}\int_0^r \frac{3\rho}{\pi r^2}\left(1-\left(\frac{\rho}{r}\right)^2\right)^2 \mathrm{d}r d\theta \\ &= \frac{6}{r^2}\int_0^r \left(\rho + \frac{\rho^5}{r^4} - 2\frac{\rho^3}{r^2}\right)\mathrm{d}r \\ &= \frac{6}{r^2}\left(\frac{r^2}{2} + \frac{r^6}{6r^4} - \frac{r^4}{2r^2}\right) = 1, \end{aligned}$$

which confirms Part 1. Let $z \in \mathbb{C}$ be arbitrary. The integral in (6.2.5) is equal to

$$\begin{aligned}
&\iint_{D(0,R)} f(z-\zeta)\lambda_r(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\qquad = \int_0^r \frac{3\rho}{\pi r^2}\left(1-\left(\frac{\rho}{r}\right)^2\right)^2 \int_0^{2\pi} f\big(z-\rho \mathrm{e}^{\mathrm{i}\theta}\big) d\theta\,\mathrm{d}r \\
&\qquad = 2\pi f(z)\int_0^r \frac{3\rho}{\pi r^2}\left(1-\left(\frac{\rho}{r}\right)^2\right)^2 \mathrm{d}r = f(z)
\end{aligned}$$

by the mean-value property (Lemma 3.4.1), proving Part 6. For $z \in \mathbb{D}$, we have

$$\begin{aligned}
\|\boldsymbol{\nabla}\lambda(z)\| &= 2(1-|z|^2)\big\|\boldsymbol{\nabla}(|z|^2)\big\| \\
&= 2(1-|z|^2)2|z|\left\|\boldsymbol{\nabla}\sqrt{x^2+y^2}\right\| = 4(1-|z|^2)|z|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\boldsymbol{\nabla}\lambda_r(z)\| &= \frac{3}{\pi r^2}\left\|\boldsymbol{\nabla}\left(\lambda\left(\frac{z}{r}\right)\right)\right\| = \frac{3}{\pi r^2}\left\|(\boldsymbol{\nabla}\lambda)\left(\frac{z}{r}\right)\right\|\frac{1}{r} \\
&= \frac{12}{\pi r^3}(1-|z|^2)|z| < \frac{4}{r^3},
\end{aligned}$$

which confirms Part 5. Since $\left|\frac{\partial\lambda_r}{\partial\overline{\zeta}}\right| = \left|\frac{1}{2}\left(\frac{\partial\lambda_r}{\partial\xi} + \mathrm{i}\frac{\partial\lambda_r}{\partial\eta}\right)\right| = \frac{1}{2}\|\boldsymbol{\nabla}\lambda_r(\zeta)\| < \frac{2}{r^3}$, we have

$$\iint_{\mathbb{C}}\left|\frac{\partial\lambda_r}{\partial\overline{\zeta}}\right|\mathrm{d}\xi\,\mathrm{d}\eta = \iint_{D(0,r)}\left|\frac{\partial\lambda_r}{\partial\overline{\zeta}}\right|\mathrm{d}\xi\,\mathrm{d}\eta < \pi r^2\frac{2}{r^3} = \frac{2\pi}{r}$$

since $\operatorname{supp}(\lambda_r) = \overline{D(0,r)}$ which verifies the inequality in Part 4.

The Part 3 is also true since

$$\begin{aligned}
\iint_{\mathbb{C}}\frac{\partial\lambda_r}{\partial\overline{\zeta}}\,\mathrm{d}\xi\,\mathrm{d}\eta &= \frac{1}{2}\int_{-r}^{r}\int_{-r}^{r}\frac{\partial\lambda_r}{\partial\xi}\,\mathrm{d}\xi\,\mathrm{d}\eta + \frac{\mathrm{i}}{2}\int_{-r}^{r}\int_{-r}^{r}\frac{\partial\lambda_r}{\partial\eta}\,\mathrm{d}\eta\,\mathrm{d}\xi \\
&= \frac{1}{2}\int_{-r}^{r}[\lambda_r(r+\mathrm{i}\eta) - \lambda(-r+\mathrm{i}\eta)]\,\mathrm{d}\eta \\
&\quad + \frac{\mathrm{i}}{2}\int_{-r}^{r}[\lambda_r(\xi+\mathrm{i}r) - \lambda_r(\xi-\mathrm{i}r)]\,\mathrm{d}\xi = 0.
\end{aligned}$$

Trivially, λ_r is continuous on $D(0,r)$ and $\mathbb{C}\setminus\overline{D(0,r)}$. Thus, we only need to prove the joint continuity of λ (the continuity of λ_r implies that of λ) on an open neighborhood of $\partial D(0,r)$.

Let $\lambda(x,y) = \left(1-x^2-y^2\right)^2$. By simple calculation, we have

$$\frac{\partial \lambda}{\partial x} = -4(1-x^2-y^2)x, \quad \frac{\partial \lambda}{\partial y} = -4(1-x^2-y^2)y.$$

At $x^2+y^2=1$, both partial derivatives vanish, and hence, they match the vanishing derivative on the complement of $\operatorname{supp}(\lambda)$, completing the proof of Part 2. □

Theorem 6.2.1 *(Tietze–Urysohn–Brouwer)*: Let $K \subseteq \mathbb{C}$ be compact and $f : K \to \mathbb{R}$ be continuous. Then $\exists g \in C^0(\mathbb{C})$ such that $g \equiv f$ on K.

Proof: For any two disjoint closed $A, B \subseteq \mathbb{C}$, consider the continuous separation function

$$\eta_{A,B}(z) = \frac{\operatorname{dist}(z,A) - \operatorname{dist}(z,B)}{\operatorname{dist}(z,A) + \operatorname{dist}(z,B)}$$

so that $\eta_{A,B}(A) = \{-1\}$ and $\eta_{A,B}(B) = \{1\}$.

For simplicity, by the boundedness of f, we may assume that $f(K) = [-1,1]$ (by a scaling and shift). We now aim to construct a sequence $\{g_n\}_{n\in\mathbb{N}_{\geq 0}}$ inductively such that

$$|g_n| \leq \frac{2^n}{3^{n+1}} \text{ on } \mathbb{C}, \quad \left| f - \sum_{k=0}^{n} g_k \right| \leq \left(\frac{2}{3}\right)^{n+1} \text{ on } K \quad \forall n \in \mathbb{N}.$$

In the case that $n=0$, define the disjoint closed sets

$$A_0 = \left\{ z \in K : f(z) \leq -\frac{1}{3} \right\} \quad \text{and} \quad B_0 = \left\{ z \in K : f(z) \geq \frac{1}{3} \right\}.$$

Let $g_0(z) = \frac{1}{3}\eta_{A_0,B_0}(z)$. It is clear that $|g_0| \leq \frac{1}{3}$ on $\mathbb{C}$. If $z \in A_0$, then $-1 \leq f(z) \leq -\frac{1}{3}$, $g_0(z) = -\frac{1}{3}$, and hence $|f - g_0| \leq \frac{2}{3}$. If $z \in B_0$, then $\frac{1}{3} \leq f(z) \leq 1$, $g_0(z) = \frac{1}{3}$, and thus $|f - g_0| \leq \frac{2}{3}$. If $z \notin A_0 \cup B_0$, then $-\frac{1}{3} < f(z) < \frac{1}{3}$ and $|f - g_0| \leq |f| + |g_0| < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. Thus, $\forall z \in K$,

$$|f(z) - g_0(z)| \leq \frac{2}{3}.$$

This proves the base case. For the inductive step, assume the claim holds for each $g_0, g_1, \ldots, g_{n-1}$. Define

$$h_n(z) = f(z) - \sum_{k=0}^{n-1} g_k(z)$$

for $z \in K$. By the inductive hypothesis, we have $|h_n| \leq \left(\frac{2}{3}\right)^n$ on K. Define the disjoint closed sets

$$A_n = \left\{z \in K : -\frac{2^n}{3^n} \leq h_n(z) \leq -\frac{2^n}{3^{n+1}}\right\}$$

and

$$B_n = \left\{z \in K : \frac{2^n}{3^n} \geq h_n(z) \geq \frac{2^n}{3^{n+1}}\right\}.$$

Let $g_n(z) = \frac{2^n}{3^{n+1}} \eta_{A_n, B_n}(z)$, so that $|g_n| \leq \frac{2^n}{3^{n+1}}$ on $\mathbb{C}$, and

$$|h_n(z) - g_n(z)| \leq \frac{2^{n+1}}{3^{n+1}}$$

for all $z \in K$ by the same argument as in the base case. Hence,

$$\left|f(z) - \sum_{k=0}^{n} g_k(z)\right| = |h_n(z) - g_n(z)| \leq \left(\frac{2}{3}\right)^{n+1}$$

for all $z \in K$, completing the induction. Because

$$|g(z)| \leq \sum_{n=0}^{\infty} |g_n(z)| \leq \frac{1}{3} \sum_{n=0}^{\infty} \frac{2^n}{3^n} = 1 \quad \forall z \in \mathbb{C},$$

the Weierstrass M–Test (Theorem 2.3.2) implies that the series $\sum_{n=0}^{\infty} g_n(z)$ converges uniformly on $\mathbb{C}$ to g. Since each g_n is continuous, Theorem 2.3.5 gives the continuity of g on $\mathbb{C}$. Finally, for any $z \in K$, we have

$$|f(z) - g(z)| \leq \lim_{n \to \infty} \frac{2^{n+1}}{3^{n+1}} = 0. \qquad \square$$

Corollary 6.2.1.1: If $K \subseteq \mathbb{C}$ is compact and $f : K \to \mathbb{C}$ is continuous, then $\exists g \in C^0(\mathbb{C})$ such that $g \equiv f$ on K and has compact support.

Proof: Let $f = u + \mathrm{i}v$ where $u, v : K \to \mathbb{R}$ are continuous. By Tietze–Urysohn–Brouwer (Theorem 6.2.1), $\exists \tilde{u}, \tilde{v} \in C^0(\mathbb{C})$ such that $\tilde{u} \equiv u$ and $\tilde{v} \equiv v$ on K. Let $R > 0$ be such that $K \subset D(0, R)$, provided by compactness. Define the piecewise-linear function

$$\psi(z) = \begin{cases} 1 & |z| \leq R, \\ 2 - \frac{|z|}{R} R < |z| < 2R, \\ 0 & |z| \geq 2R \end{cases}$$

such that $\psi \in C^0(\mathbb{C})$ and is compactly supported. Let $g(z) = (\tilde{u}(z) + \mathrm{i}\tilde{v}(z))\psi(z)$, and the assertion follows. $\square$

Let $f \in C^0(K)$ be holomorphic on $\overset{\circ}{K}$. Then f has a continuous extension to all of $\mathbb{C}$ by virtue of Corollary 6.2.1.1. Define the *modulus of continuity* of f to be the function $\omega_f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with

$$\omega_f(\delta) = \sup_{\substack{z,\zeta \in \mathbb{C} \\ |z-\zeta| \leq \delta}} |f(z) - f(\zeta)|.$$

Because f has compact support, it must be uniformly continuous; hence we have $\lim_{\delta \to 0^+} \omega_f(\delta) = 0$.

For $r > 0$, define

$$\Phi(z) = \iint_{\mathbb{C}} \lambda_r(z - \zeta) f(\zeta) \, \mathrm{d}\xi \, \mathrm{d}\eta \quad \text{where} \quad \zeta = \xi + \mathrm{i}\eta, \qquad (6.2.6)$$

where λ_r employs the same definition as in (6.2.4).

Proposition 6.2.4: The function Φ as in (6.2.6) satisfies:
1 $\Phi \in C^1(\mathbb{C})$ and has compact support.
2 $\Phi \equiv f$ on $U = \{z \in K : \operatorname{dist}(z, \mathbb{C} \setminus K) > r\}$.
3 $|f(z) - \Phi(z)| \leq \omega_f(r)$ for all $z \in \mathbb{C}$.
4 For all $z \in \mathbb{C}$, $\left|\frac{\partial \Phi}{\partial \bar{z}}(z)\right| \leq \frac{4\pi \omega_f(r)}{r}$.
5 $\Phi(z) = -\frac{1}{\pi} \iint_H \frac{\partial \Phi}{\partial \bar{\zeta}}(\zeta) \frac{\mathrm{d}\xi \, \mathrm{d}\eta}{\zeta - z}$ for $z \in \mathbb{C}$, where $H = \operatorname{supp}(\Phi) \setminus U$.

Proof: Because $\operatorname{supp}(\lambda_r(z - \zeta)) = \overline{D(z, r)}$ and $\operatorname{supp} f$ is compact, for sufficiently large z, the two supports will be disjoint and hence the integrand vanishes for all ζ. We can explicitly find that

$$\begin{aligned} \frac{\partial \Phi}{\partial x} &= \lim_{\substack{\Delta x \to 0 \\ \Delta x \in \mathbb{R}}} \frac{\Phi(z + \Delta x) - \Phi(\Delta x)}{\Delta x} \\ &= \lim_{\substack{\Delta x \to 0 \\ \Delta x \in \mathbb{R}}} \int_{\mathbb{C}} \frac{\lambda_r(z + \Delta x - \zeta) - \lambda_r(z - \zeta)}{\Delta x} f(\zeta) \, \mathrm{d}\xi \wedge \mathrm{d}\eta. \end{aligned}$$

Because f is continuous and vanishes on a compact set, it is bounded. Similarly, Part 2 of Proposition 6.2.3 implies that $\frac{\partial \lambda_r}{\partial x}$ is bounded. Hence, by Lebesgue's Dominated Convergence Theorem, we have

$$\frac{\partial \Phi}{\partial x} = \int_{\mathbb{C}} \frac{\partial \lambda_r}{\partial x}(z - \zeta) f(\zeta) \, \mathrm{d}\xi \wedge \mathrm{d}\eta,$$

and similarly,

$$\frac{\partial \Phi}{\partial y} = \int_{\mathbb{C}} \frac{\partial \lambda_r}{\partial y}(z - \zeta) f(\zeta) \, \mathrm{d}\xi \wedge \mathrm{d}\eta.$$

Hence, $\Phi \in C^1(\mathbb{C})$ and this is Part 1. Because

$$\Phi(z) = \iint_{\mathbb{C}} \lambda_r(z-\zeta) f(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta = \iint_{\mathbb{C}} \lambda_r(\zeta) f(z-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

by Part 1 of Proposition 6.2.3, we have

$$\begin{aligned} |f(z)-\Phi(z)| &\le \left| \int_{\mathbb{C}} f(z)\lambda_r(\zeta)\,\mathrm{d}\xi\wedge\mathrm{d}\eta - \int_{\mathbb{C}} f(z-\zeta)\lambda_r(\zeta)\,\mathrm{d}\xi\wedge\mathrm{d}\eta \right| \\ &= \left| \int_{\mathbb{C}} \lambda_r(\zeta)(f(z)-f(z-\zeta))\,\mathrm{d}\xi\wedge\mathrm{d}\eta \right| \qquad (6.2.7) \\ &\le \int_{D(0,r)} \lambda_r(\zeta)|f(z)-f(z-\zeta)|\,\mathrm{d}\xi\wedge\mathrm{d}\eta \le \omega_f(r), \end{aligned}$$

which implies Part 3. For $z \in U$, $\zeta \in D(0,r)$ now implies that $z-\zeta \in \overset{\circ}{K}$ and hence $f(z)-f(z-\zeta)$ is holomorphic in ζ on $D(z,r)$. By Part 6 of Proposition 6.2.3, (6.2.7) becomes

$$\left| \int_{\mathbb{C}} \lambda_r(\zeta)(f(z)-f(z-\zeta))\,\mathrm{d}\xi\wedge\mathrm{d}\eta \right| = |f(z)-f(z-0)| = 0,$$

which proves Part 2. Because $\forall z \in \mathbb{C}$,

$$\begin{aligned} \frac{\partial\Phi}{\partial\bar{z}}(z) &= \frac{1}{2}\left(\frac{\partial\Phi}{\partial x} + \mathrm{i}\frac{\partial\Phi}{\partial y}\right) = \int_{\mathbb{C}} \frac{\partial\lambda_r}{\partial\bar{z}}(z-\zeta) f(\zeta)\,\mathrm{d}\xi\wedge\mathrm{d}\eta \\ &= \int_{\mathbb{C}} \frac{\partial\lambda_r}{\partial\bar{\zeta}}(\zeta) f(z-\zeta)\,\mathrm{d}\xi\wedge\mathrm{d}\eta \\ &= \int_{\mathbb{C}} \frac{\partial\lambda_r}{\partial\bar{\zeta}}(\zeta) f(z-\zeta)\,\mathrm{d}\xi\wedge\mathrm{d}\eta - f(z)\int_{\mathbb{C}} \frac{\partial\lambda_r}{\partial\bar{\zeta}}\,\mathrm{d}\xi\wedge\mathrm{d}\eta \\ &= \int_{\mathbb{C}} \frac{\partial\lambda_r}{\partial\bar{\zeta}}(\zeta)(f(z-\zeta)-f(z))\,\mathrm{d}\xi\wedge\mathrm{d}\eta \end{aligned}$$

by Part 3 of Proposition 6.2.3. Hence,

$$\begin{aligned} \left|\frac{\partial\Phi}{\partial\bar{z}}\right| &\le \iint_{D(0,r)} \left|\frac{\partial\lambda_r}{\partial\bar{\zeta}}\right| |f(z-\zeta)-f(z)|\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\le \omega_f(r) \iint_{D(0,r)} \|\boldsymbol{\nabla}\lambda_r\|\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\le \frac{4\omega_f(r)}{r^3} \iint_{D(0,r)} \mathrm{d}\xi\,\mathrm{d}\eta \le \frac{4\omega_f(r)}{r^3}\cdot\pi r^2 = \frac{4\pi\omega_f(r)}{r}, \end{aligned}$$

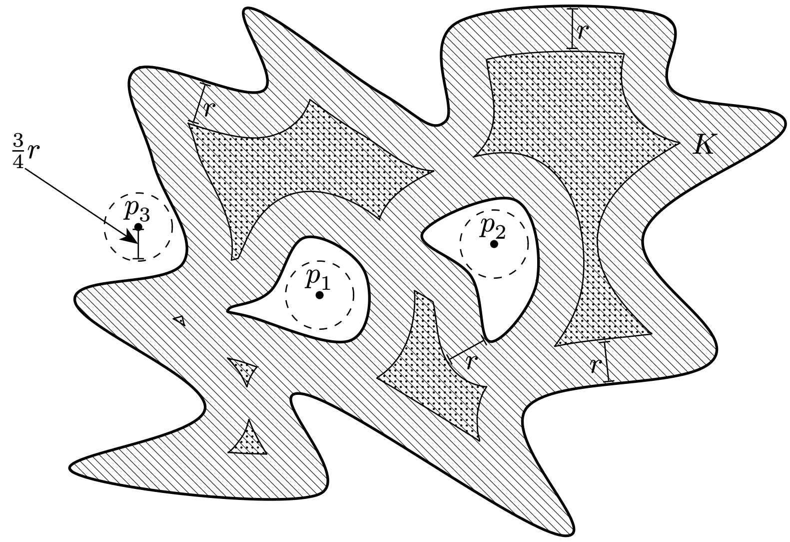

Figure 15: The striped region bounds K, while the thin lines bound the dotted region U.

by Part 5 of Proposition 6.2.3, confirming Part 4. Finally, Part 5 follows from Corollary 3.1.3.1 (since outside the support the integral trivially vanishes and within U, $\frac{\partial \Phi}{\partial \overline{\zeta}}$ vanishes as a consequence of holomorphy). $\square$

Theorem 6.2.2 *(Mergelyan)*: Let $K \subseteq \mathbb{C}$ be compact such that $\hat{\mathbb{C}} \setminus K$ has finitely many connected components. Let $E \subseteq \hat{\mathbb{C}} \setminus K$ contain exactly one point from each of the connected components of $\hat{\mathbb{C}} \setminus K$. Suppose $f \in C^0(K)$ is holomorphic on $\overset{\circ}{K}$. Then $\forall \varepsilon > 0$, there exists a rational function $\psi(z)$ with poles in E such that

$$\sup_{z \in K} |\psi(z) - f(z)| < \varepsilon.$$

Proof: Let $F = \{p_k\}_{1 \leq k \leq n}$ contain precisely one point from each connected component of $\hat{\mathbb{C}} \setminus K$ (such that each $p_k \neq \infty$ is finite). Suppose that r is chosen such that $0 < \frac{3}{4}r < \operatorname{dist}(K, F)$ so that for each $p_k \in F$ not equal to ∞,

$$\overline{D\left(p_k, \frac{3}{4}r\right)} \subset \hat{\mathbb{C}} \setminus K.$$

Define the extension of f, Φ, U, and H as in the previous results (see Figure 15). Hence, (see Figure 16)

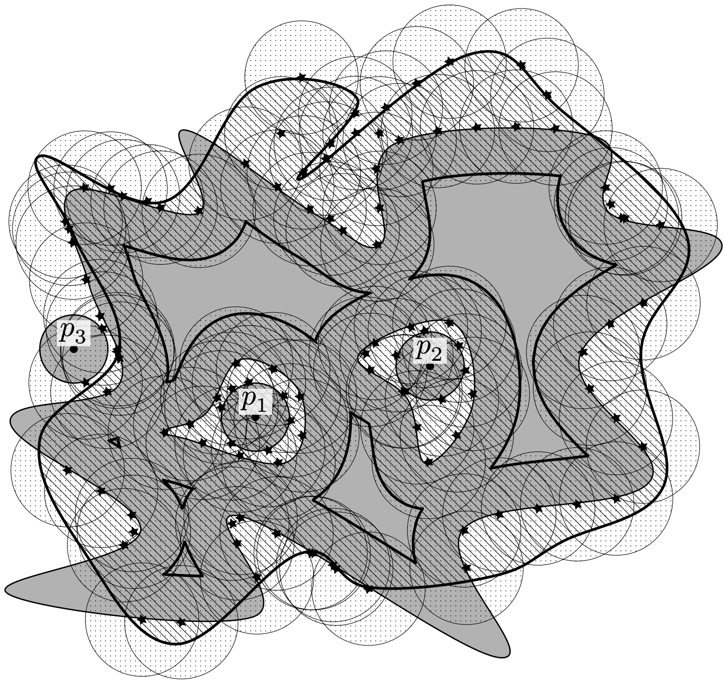

Figure 16: The striped region represents H while the unshaded regions represent the set which ζ_k can be in, each of which are denoted by small stars. The dotted disks represent the finite subcover of H: notice that every striped region is also dotted.

$$\left\{D\left(\zeta_k,\frac{5}{4}r\right):(\forall)\zeta_k\in\hat{\mathbb{C}}\setminus\left(K\cup\overline{D\left(p_k,\frac{3}{4}r\right)}\right),1\le k\le n\right\}$$

covers a (compact) r-neighborhood of $\hat{\mathbb{C}}\setminus K$ (so that each $\zeta_k\notin K$, and is labeled so that each ζ_k is in the same connected component as p_k) (in the case that $p_k=\infty$, let the disk inside be the empty set). Thus, the collection also covers H. A finite subcover $\left\{D\left(\zeta_k^{(j)},\frac{5}{4}r\right)\right\}_{\substack{1\le j\le m_k\\ 1\le k\le n}}$ covering H exists by the Heine–Borel Theorem (Theorem 1.1.3).

By the connectivity of each component of $\hat{\mathbb{C}}\setminus K$, there exists a piecewise-linear simple curve $\gamma_k^{(j)}$ for all $1\le k\le n$, $1\le j\le m_k$, joining $\zeta_k^{(j)}$ and p_k, which lies entirely within $\hat{\mathbb{C}}\setminus K$. The compact disks $\overline{D\left(\zeta_k^{(j)},\frac{3}{4}r\right)}$ are all disjoint from their corresponding p_k since each $\zeta_k^{(j)}\notin\overline{D\left(p_k,\frac{3}{4}r\right)}$ by definition.

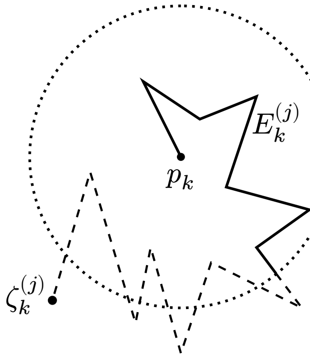

Figure 17: The construction of E_k^j. The entire polyline from p_k to $\zeta_k^{(j)}$ is $\gamma_k^{(j)}$

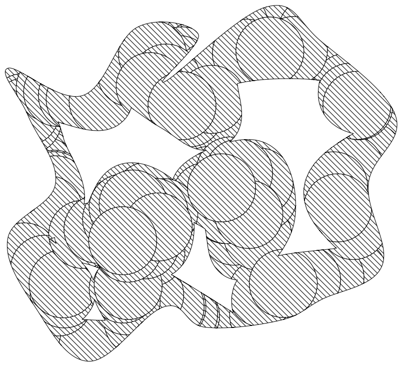

Figure 18: A conceptual construction of $\left\{H_k^{(j)}\right\}$, which unions to H.

Hence, the intersection $\overline{D\left(\zeta_k^{(j)}, \frac{3}{4}r\right)} \cap \gamma_k^{(j)}$ consists of at least one connected component joining $\zeta_k^{(j)}$ to a point on $\partial D\left(\zeta_k^{(j)}, \frac{3}{4}r\right)$. Denote the connected component of this intersection by $E_k^{(j)}$, satisfying $\operatorname{diam} E_k^{(j)} \geq \frac{3}{4}r > \frac{r}{2}$ and $E_k^{(j)} \cap K = \varnothing$ (see Figure 17).

Now for each j and k, Proposition 6.2.2 now provides the existence of a family of holomorphic functions $\varphi_{\zeta,k}^{(j)} : \hat{\mathbb{C}} \setminus E_k^{(j)} \to \mathbb{C}$ given with $\zeta \in D\left(\zeta_k^{(j)}, \frac{5}{4}r\right)$ such that

$$\left|\varphi_{\zeta,k}^{(j)}(z)\right| \leq \frac{584}{r}, \quad \left|\varphi_{\zeta,k}^{(j)}(z) - \frac{1}{z-\zeta}\right| \leq \frac{4676}{|z-\zeta|^3}, \quad \forall z \in \hat{\mathbb{C}} \setminus E_k^{(j)} \tag{6.2.8}$$

Let $\tilde{H}_k^{(j)} = H \cap D\left(\zeta_k^{(j)}, \frac{5}{4}r\right)$, for each j, k and construct the disjoint sets

$$H_k^{(j)} = \tilde{H}_k^{(j)} \setminus \left(\bigcup_{j'<j} \tilde{H}_k^{(j')} \cup \bigcup_{k'<k} \bigcup_{j' \leq m_{k'}} \tilde{H}_{k'}^{(j')}\right) \text{ if } j \neq 1, \quad H_1^{(1)} = \tilde{H}_1^{(1)}.$$

as done in Figure 18. Thus the union

$$\bigcup_{k=1}^{n} \bigcup_{j=1}^{m_k} H_k^{(j)} = H \cap \left(\bigcup D\left(\zeta_k^{(j)}, \frac{5}{4}r\right)\right) = H$$

since the set of all $D\left(\zeta_k^{(j)}, \frac{5}{4}r\right)$ covers H. Let

$$\Psi(z) = \frac{1}{\pi} \sum_{k=1}^{n} \sum_{j=1}^{m_k} \int_{H_k^{(j)}} \frac{\partial \Phi}{\partial \bar{\zeta}} \varphi_{\zeta,k}^{(j)}(z) \, \mathrm{d}\xi \wedge \mathrm{d}\eta,$$

where $\zeta = \xi + \mathrm{i}\eta$, $\forall z \notin \bigcup E_k^{(j)}$. Because

$$\frac{\Psi(z+\Delta x) - \Psi(z)}{\Delta x} = \frac{1}{\pi} \sum_{k=1}^{n} \sum_{j=1}^{m_k} \int_{H_k^{(j)}} \frac{\partial \Phi}{\partial \overline{\zeta}} \frac{\varphi_{\zeta,k}^{(j)}(z+\Delta x) - \varphi_{\zeta,k}^{(j)}(z)}{\Delta x} \,\mathrm{d}\xi \wedge \mathrm{d}\eta,$$

and both $\partial\Phi/\partial\overline{\zeta}$ and the integrand is continuous on a set (we only need to consider the factors involving $\varphi_{\zeta,k}^{(j)}$ since $\partial\Phi/\partial\overline{\zeta}$ is independent from z) by Cauchy's Estimates and the first bound of (6.2.8), Lebesgue's Dominated Convergence gives that

$$\frac{\partial \Psi}{\partial x} = \frac{1}{\pi} \sum_{k=1}^{n} \sum_{j=1}^{m_k} \int_{H_k^{(j)}} \frac{\partial \Phi}{\partial \overline{\zeta}} \frac{\partial \varphi_{\zeta,k}^{(j)}}{\partial x}(z) \,\mathrm{d}\xi \wedge \mathrm{d}\eta,$$

and in analogous fashion,

$$\frac{\partial \Psi}{\partial y} = \frac{1}{\pi} \sum_{k=1}^{n} \sum_{j=1}^{m_k} \int_{H_k^{(j)}} \frac{\partial \Phi}{\partial \overline{\zeta}} \frac{\partial \varphi_{\zeta,k}^{(j)}}{\partial y}(z) \,\mathrm{d}\xi \wedge \mathrm{d}\eta.$$

Hence, Ψ is holomorphic on $\hat{\mathbb{C}} \setminus \bigcup_{k=1}^{n} \bigcup_{j=1}^{m_k} E_k^{(j)}$, a neighborhood of K. Since $\forall z \in \hat{\mathbb{C}} \setminus \bigcup E_k^{(j)}$, by Part 2 of Proposition 6.2.2,

$$\begin{aligned}
|\Psi(z) - \Phi(z)| &= \left| \frac{1}{\pi} \sum_{k=1}^{n} \sum_{j=1}^{m_k} \iint_{H_k^{(j)}} \frac{\partial \Phi}{\partial \overline{\zeta}} \varphi_{\zeta,k}^{(j)}(z) \,\mathrm{d}\xi \,\mathrm{d}\eta - \frac{1}{\pi} \iint_{H} \frac{\partial \Phi}{\partial \overline{\zeta}} \frac{\mathrm{d}\xi \,\mathrm{d}\eta}{z-\zeta} \right| \\
&= \frac{1}{\pi} \left| \sum_{k=1}^{n} \sum_{j=1}^{m_k} \iint_{H_k^{(j)}} \frac{\partial \Phi}{\partial \overline{\zeta}} \left(\varphi_{\zeta,k}^{(j)}(z) - \frac{1}{z-\zeta} \right) \mathrm{d}\xi \,\mathrm{d}\eta \right| \\
&\leq \frac{1}{\pi} \sum_{k=1}^{n} \sum_{j=1}^{m_k} \iint_{H_k^{(j)}} \left| \frac{\partial \Phi}{\partial \overline{\zeta}} \right| \left| \varphi_{\zeta,k}^{(j)}(z) - \frac{1}{z-\zeta} \right| \mathrm{d}\xi \,\mathrm{d}\eta \\
&\leq \frac{1}{\pi} \sum_{k=1}^{n} \sum_{j=1}^{m_k} \left(\iint_{H_k^{(j)} \cap D(z,2r)} + \iint_{H_k^{(j)} \setminus D(z,2r)} \right) \\
&\qquad \cdot \left| \frac{\partial \Phi}{\partial \overline{\zeta}} \right| \left| \varphi_{\zeta,k}^{(j)}(z) - \frac{1}{z-\zeta} \right| \mathrm{d}\xi \,\mathrm{d}\eta
\end{aligned}$$

The estimates in Part 4 of Proposition 6.2.4, in tandem with those from (6.2.8) now give that

$$
\begin{aligned}
|\Psi(z)-\Phi(z)| &\leq 18704 r\omega_f(r) \sum_{k=1}^{n}\sum_{j=1}^{m_k} \iint_{H_k^{(j)}\setminus D(z,2r)} \frac{1}{|z-\zeta|^3}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad + \frac{4\omega_f(r)}{r} \sum_{k=1}^{n}\sum_{j=1}^{m_k} \iint_{H_k^{(j)}\cap D(z,2r)} \left(\frac{584}{r}+\frac{1}{|z-\zeta|}\right)\mathrm{d}\xi\,\mathrm{d}\eta \\
&\leq 18704 r\omega_f(r) \iint_{|\zeta|>2r} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{|\zeta|^3} \\
&\quad + \frac{4\omega_f(r)}{r} \iint_{|\zeta|<2r} \left(\frac{584}{r}+\frac{1}{|\zeta|}\right)\mathrm{d}\xi\,\mathrm{d}\eta
\end{aligned}
$$

through a linear change of variables. Now evaluation via polar coordinates (with $\rho e^{i\theta} = \xi + i\eta$, $\mathrm{d}\xi \wedge \mathrm{d}\eta = \rho\,\mathrm{d}r \wedge d\theta$) yields a revised upper bound of

$$
\begin{aligned}
&18704 r\omega_f(r) \int_0^{2\pi}\int_{2r}^{\infty} \frac{\mathrm{d}r\,\mathrm{d}\theta}{\rho^2} \\
&\quad + \frac{4\omega_f(r)}{r}\left(\frac{584}{r}\operatorname{area} D(0,2r) + \int_0^{2\pi}\int_0^{2r} \mathrm{d}r d\theta\right) \\
&= 18704 r\omega_f(r) 2\pi \left[\frac{1}{\rho}\right]_{\infty}^{2r} + \frac{9344\pi\omega_f(r) r^2}{r^2} + \frac{4\omega_f(r)}{r} 4r\pi \\
&= 18704\pi\omega_f(r) + 9344\pi\omega_f(r) + 16\pi\omega_f(r) = 28064\pi\omega_f(r).
\end{aligned}
$$

Runge's Theorem (Theorem 6.1.1) provides the existence of some rational function ψ with poles in E such that

$$
\sup_{z\in K}|\psi(z)-\Psi(z)| \leq \pi\omega_f(r)
$$

since Ψ is holomorphic on a neighborhood of K (to assure this bound is positive, we assume f is not identically zero, otherwise the assertion is trivial). Therefore, for all $z \in K$, we have (the third supremum term coming from Part 3 of Proposition 6.2.4)

$$
\begin{aligned}
\sup_{z\in K}|\psi(z)-f(z)| &\leq \sup_{z\in K}|\psi(z)-\Psi(z)| + |\Psi(z)-\Phi(z)| + |\Phi(z)-f(z)| \\
&\leq 28065\pi\omega_f(r) + \omega_f(r) \leq 28066\pi\omega_f(r).
\end{aligned}
$$

Because $\lim_{r\to 0^+}\omega_f(r) = 0$, for any $\varepsilon > 0$, there exists a $r > 0$ such that

$$
\omega_f(r) < \frac{\varepsilon}{28066\pi}.
$$

Hence for any such ε, we now construct ψ in accordance with an r satisfying $28066\pi\omega_f(r) < \varepsilon$. □

6.3 Analytic Capacity

The theory of rational approximation is essentially built upon the concept of *analytic capacity*, which was introduced in 1940 by Finnish mathematician Lars Ahlfors. Our purpose here is to give a brief and elementary introduction. Despite its importance, still many trivially simple results remain conjecture.

The uses of analytic capacity are present in many other topics of complex analysis. Analytic capacity serves as a natural framework for general rational approximation theory. Our purpose here is to hint at how analytic capacity theory relates to the proof of Theorem 6.2.2 and pertinent problems in general.

Definition 6.3.1 *(Analytic Capacity)*: Let $K \subseteq \mathbb{C}$ be compact. The *analytic capacity* of K is defined as

$$\gamma(K) = \sup\left\{ |f'(\infty)| : \begin{bmatrix} f \text{ is holomorphic on } \hat{\mathbb{C}} \setminus K \\ f\left(\hat{\mathbb{C}} \setminus K\right) \subseteq \overline{\mathbb{D}} \\ f(\infty) = 0 \end{bmatrix} \right\}, \quad (6.3.1)$$

where $f'(\infty)$ is defined as in (6.3.1). For an arbitrary set $U \subseteq \mathbb{C}$, we define

$$\sup\{\gamma(K) : K \subseteq U \land K \text{ is compact}\}.$$

Intuitively, γ measures the extent to which bounded analytic functions outside K can deviate from constancy. Generally, the "larger" K is, the greater the capacity is.

Proposition 6.3.1: If $K \subset \mathbb{C}$ is a compact set of discrete points, then $\lambda(K) = 0$.

Proof: For any $f : \hat{\mathbb{C}} \setminus K \to \mathbb{C}$ holomorphic with $f\left(\hat{\mathbb{C}} \setminus K\right) \subseteq \overline{\mathbb{D}}$, since f is bounded, the Riemann's Theorem for removable singularities (Theorem 3.2.6) allows for an analytic continuation onto all of $\hat{\mathbb{C}}$. Then Liouville's Theorem (Theorem 3.2.3) implies that f is constant and $f'(\infty) = 0$. Hence $\gamma(K) = 0$.□

Theorem 6.3.1: For $K_1 \subseteq K_2$ both compact in $\mathbb{C}$, $\gamma(K_1) \leq \gamma(K_2)$.

Proof: This follows directly from the definition and the fact that any function holomorphic on $\hat{\mathbb{C}} \setminus K_1$ is also holomorphic on $\hat{\mathbb{C}} \setminus K_2$. □

The preceding results above hint at the monotonous behavior of capacity. However, currently it is not known whether a general *subadditivity* property holds for analytic capacity, or that

$$\gamma(K_1 \cup K_2) \overset{?}{\leq} \gamma(K_1) + \gamma(K_2).$$

Recent results hint the affirmative, as many special cases of the relation have been proved; the question of subadditivity has been proved in the affirmative for disjoint compact continua, and recent findings by Xavier Tolsa show that capacity is (countably) semi-(sub)additive (the existence of an absolute constant C such that $\gamma(K_1 \cup K_2) \leq C[\gamma(K_1) + \gamma(K_2)]$).

We now give some quantifying examples of how analytic capacity measures a type of "largeness" of compact sets, (rather much like area, which satisfies the subadditivity relation). First we define a specific classification of compact sets.

An alternative perspective of this "largeness" pertains to a certain *removability* of sets. A compact set $K \subset \mathbb{C}$ is considered to be *removable* if every bounded holomorphic function on the complement can be extended to $\mathbb{C}$. For instance, the analytic capacity $\gamma(\{a\})$ of any singleton $\{a\}$ (any singular point) or set of discrete points is 0, as evidenced by Proposition 6.3.1; and moreover, any singleton or compact set of discrete points is a removable set. In a heuristic sense, analytic capacity measures the irremovability of a set, and larger sets tend to be "less removable."

A compact set $K \subset \mathbb{C}$ is a *continuum* if it is connected, $\mathbb{C} \setminus K$ is connected, and if it is not a singleton (K contains at least 2 distinct points).

Proposition 6.3.2: Let $K \subset \mathbb{C}$ be a continuum. Then $\gamma(K) = |f'(\infty)|$ where $f : \hat{\mathbb{C}} \setminus K \to \mathbb{D}$ is a biholomorphism satisfying $f(\infty) = 0$ (i. e. the maximal $|f'(\infty)|$ in the supremum of the definition of analytic capacity is attained when f is biholomorphic).

Proof: Let f be the biholomorphism, $g : \hat{\mathbb{C}} \setminus K \to \mathbb{D}$ be holomorphic (not necessarily surjective) mapping ∞ to 0. Since $h \equiv g \circ f^{-1} : \mathbb{D} \to \mathbb{D}$ and maps 0 to 0, the Schwarz Lemma (Lemma 3.5.1) implies that

$$|h(z)| \leq |z|$$

for all $z \in \mathbb{D}$. Thus, $|g(z)| \leq |f(z)|$, and

$$|g'(\infty)| = \lim_{z \to \infty} |zg(z)| \leq \lim_{z \to \infty} |zf(z)| = |f'(\infty)|. \qquad \square$$

Proposition 6.3.3: The analytic capacity of any closed disk is the radius.

Proof: Since $\overline{D(a, r)}$ is a continuum, a biholomorphism $f : \hat{\mathbb{C}} \setminus \overline{D(a, r)} \to \mathbb{D}$ such that $f(\infty) = 0$. One such biholomorphism is given by

$$f(z) = \frac{r}{z-a}, \quad f'(\infty) = \lim_{z \to 0} \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{r}{\frac{1}{z} - a} \right) = r.$$

Hence, Proposition 6.3.2, gives that $\gamma\left(\overline{D(a,r)}\right) = r$. □

Proposition 6.3.4: If $K \subset \mathbb{C}$ is a continuum, then

$$\frac{\operatorname{diam} K}{4} \le \gamma(K) \le \operatorname{diam} K.$$

Proof: Assume $f : \hat{\mathbb{C}} \setminus K \to \mathbb{D}$ is a biholomorphism mapping ∞ to 0. The lower bound follows directly from Proposition 6.2.1. Let $p \in K$ be arbitrary, then for any $q \in K$, we obtain $|p - q| \le \operatorname{diam} K$, implying that $K \subseteq \overline{D(p, \operatorname{diam} K)}$. By Proposition 6.3.3, we have $\gamma\left(\overline{D(p, \operatorname{diam} K)}\right) = \operatorname{diam} K$, and Theorem 6.3.1 consequently gives the upper bound of

$$\gamma(K) \le \operatorname{diam} K.$$

□

We outline the precise connections to rational approximation:

Theorem 6.3.2: Let $K \subseteq \mathbb{C}$ be compact such that $\exists c > 0$ such that $\forall \delta > 0$, $\forall p \in \partial K$,

$$\gamma(D(p, \delta) \setminus K) \ge c\delta.$$

Then $\forall f \in C^0(K)$ holomorphic on $\overset{\circ}{K}$ can be uniformly and rationally approximated on K with poles in $\hat{\mathbb{C}} \setminus K$.

Corollary 6.3.2.1: Let $K \subseteq \mathbb{C}$ be compact. If the connected components U_j of $\hat{\mathbb{C}} \setminus K$ give the uniform existence of some $\delta > 0$ such that $\forall j$, $\operatorname{diam} U_j \ge \delta$, then $\forall f \in C^0(K)$ holomorphic on $\overset{\circ}{K}$ can be uniformly and rationally approximated on K with poles in $\hat{\mathbb{C}} \setminus K$.

Notice here that no restrictions are imposed on the finiteness of the number of connected components of the complement. The general conclusion given for Mergelyan's Theorem is not true for more general compact sets.

The counterexample we now provide due to [1], we provide the construction of the compact set K.

Example 6.3.1: There exists a compact set $K \subseteq \mathbb{C}$ and $f \in C^0(K)$, such that f is holomorphic on $\overset{\circ}{K}$ and cannot be rationally approximated on K.

Proof: Let $S = \{s_k\}_{k \in \mathbb{N}}$ be a countably dense set of points in $\mathbb{D}$ (use a bijection $\mathbb{N} \to \mathbb{Q}$ and Cantor's pairing function $\mathbb{N} \to \mathbb{N}^2$ to get a surjection $\mathbb{N} \to \mathbb{Q}^2 \cap \mathbb{D}$).

Fix $0 < \varepsilon' < 1$. Let $z_1 = s_1$ and $r_1 = \frac{\varepsilon}{2} < \varepsilon'$, $r_1^2 < \frac{1}{2}$. For each $k \in \mathbb{N}$, let z_k be the first to be the first point in the dense sequence S such that

$$c_k \notin \bigcup_{j=1}^{k-1} \overline{D(z_j, r_j)}.$$

Then choose r_k such that $D(z_k, r_k)$ lies in $\mathbb{D}$ and does not intersect any previous $D(z_j, r_j)$ for $j < k$ (possible by the fact that each z_j does not lie on the boundary of the previous disks) and so that

$$0 < r_k \leq \frac{\varepsilon}{2} - \frac{1}{2}\sum_{j=1}^{k-1} r_j < \varepsilon' - \sum_{j=1}^{k-1} r_j \Longrightarrow \sum_{j=1}^{k} < \varepsilon'$$

under the inductive hypothesis that $\sum_{j=1}^{k-1} r_j < \varepsilon'$. Apply another bound, so that

$$0 < r_k^2 \leq \frac{1}{4} - \frac{1}{2}\sum_{j=1}^{k-1} r_j^2 < \frac{1}{2} - \sum_{j=1}^{k-1} r_j^2 \Longrightarrow \sum_{j=1}^{k} r_j^2 < \frac{1}{2}$$

under the additional assumption that $\sum_{j=1}^{k-1} r_j^2 < \frac{1}{2}$. Repeat this process inductively for all $k \in \mathbb{N}$. Define

$$K = \overline{\mathbb{D}} \setminus \bigcup_{j=1}^{\infty} D(z_j, r_j),$$

which is compact. For any point $z \in K$, no disk centered at z exists such that $D(z, \delta)$ is contained in K, since a subsequence of S accumulating to z in $\overline{\mathbb{D}}$ is removed from K. Hence, $\overset{\circ}{K} = \varnothing$. Hence, any $f \in C^0(K)$ is holomorphic on the interior.

(This general construction of K is known as the *Swiss cheese set*)

We now show that $z \mapsto \overline{z}$ cannot be uniformly rationally approximated on K. By explicit calculation or Green's Theorem (Theorem 3.1.2), we have

$$\begin{aligned}\oint_{\partial D(z_j, r_j)} \overline{z}\,\mathrm{d}z &= \iint_{D(z_j, r_j)} \mathrm{d}(\overline{z}\,\mathrm{d}z) = 2\mathrm{i}\iint_{D(z_j, r_j)} \mathrm{d}x \wedge \mathrm{d}y \\ &= 2\pi \mathrm{i} r_j^2 \Longrightarrow \left|\sum_{j=1}^{\infty} \oint_{\partial D(z_j, r_j)} \overline{z}\,\mathrm{d}z\right| \leq \pi,\end{aligned}$$

and by similar reasoning

$$\left|\oint_{\partial\mathbb{D}} \overline{z}\,\mathrm{d}z\right| = 2\pi \implies \left|\left(\oint_{\partial\mathbb{D}} - \sum_{j=1}^{\infty} \oint_{\partial D(z_j,r_j)}\right)\overline{z}\,\mathrm{d}z\right| \geq \pi.$$

For any rational ψ with poles off K,

$$\left(\oint_{\partial\mathbb{D}} - \sum_{j=1}^{\infty} \oint_{\partial D(z_j,r_j)}\right)\psi(z)\,\mathrm{d}z = 0$$

by Theorem 3.1.5. The summation's convergence follows from $\left|\sum \oint_{\partial D(z_j,r_j)} \psi(z)\,\mathrm{d}z\right|$ being termwise absolutely bounded by $\sum 2\pi M r_j$, which converges by construction. Now, if $|\overline{z} - \psi(z)| < \frac{1}{4}$ on K, then

$$\begin{aligned}\pi &\leq \left|\left(\oint_{\partial\mathbb{D}} - \sum_{j=1}^{\infty} \oint_{\partial D(z_j,r_j)}\right)\overline{z}\,\mathrm{d}z\right| \leq \left|\oint_{\partial\mathbb{D}} - \sum_{j=1}^{\infty} \oint_{\partial D(z_j,r_j)}\right| |\psi(z) - \overline{z}||\mathrm{d}z| \\ &\leq 2\pi\left(1 - \sum_{j=1}^{\infty} r_j\right)\frac{1}{4} \leq \frac{\pi}{2},\end{aligned}$$

which is impossible. □

7 Harmonic Functions

8 Differential Geometry

8.1 Gaussian Curvature of a Surface

We will give a brief introduction to the curvature of a surface for heuristic intuition.

Suppose $U \subseteq \mathbb{R}^2$ is a region, and let $(u, v) \in U$. Consider a surface parameterized via

$$\vec{\mathbf{r}}(u, v) = (x(u, v), y(u, v), z(u, v)) \in \mathbb{R}^3,$$

where $x, y, z \in C^2(U)$. If $\vec{\mathbf{r}}'_u \times \vec{\mathbf{r}}'_v$ never vanishes for $(u, v) \in U$, then $\vec{\mathbf{r}}(U)$ defines a smooth surface Σ. For a fixed $(u, v) \in U$, the vectors $\vec{\mathbf{r}}'_u$ and $\vec{\mathbf{r}}'_v$ form the basis of the tangent space (a plane) of Σ at $P = \vec{\mathbf{r}}(u, v)$, denoted by $T_P\Sigma = \operatorname{span}(\vec{\mathbf{r}}'_u(P), \vec{\mathbf{r}}'_v(P))$.

The square of the length of the vector infinitesimal $\mathrm{d}\vec{\mathbf{r}} = \vec{\mathbf{r}}'_u\,\mathrm{d}u + \vec{\mathbf{r}}'_v\,\mathrm{d}v$, or

$$\mathrm{I} = \mathrm{d}s^2 = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2, \tag{8.1.1}$$

is known as the *first fundamental form* of Σ, where $E = \vec{\mathbf{r}}'_u \cdot \vec{\mathbf{r}}'_u$, $F = \vec{\mathbf{r}}'_u \cdot \vec{\mathbf{r}}'_v$, and $G = \vec{\mathbf{r}}'_v \cdot \vec{\mathbf{r}}'_v$.

Let $Q = \vec{\mathbf{r}}(u + \Delta u, v + \Delta v)$ be near P. It follows that $\overrightarrow{PQ} = \vec{\mathbf{r}}(u + \Delta u, v + \Delta v) - \vec{\mathbf{r}}(u, v)$. The distance between Q and $T_P\Sigma$ is $\overrightarrow{PQ} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}} = \frac{\vec{\mathbf{r}}'_u \times \vec{\mathbf{r}}'_v}{\|\vec{\mathbf{r}}'_u \times \vec{\mathbf{r}}'_v\|}$. By application of the multivariate Taylor's Theorem, we have

$$\overrightarrow{PQ} = \vec{\mathbf{r}}'_u \Delta u + \vec{\mathbf{r}}'_v \Delta v + \frac{1}{2}\left(\vec{\mathbf{r}}''_{uu} \Delta u^2 + 2\vec{\mathbf{r}}''_{uv} \Delta u \Delta v + \vec{\mathbf{r}}''_{vv} \Delta v^2\right) + \mathcal{O}(\Delta u^3 + \Delta v^3),$$

and therefore,

$$\overrightarrow{PQ} \cdot \hat{\mathbf{n}} = \frac{1}{2}\left(\vec{\mathbf{r}}''_{uu} \cdot \hat{\mathbf{n}} \Delta u^2 + 2\vec{\mathbf{r}}''_{uv} \cdot \hat{\mathbf{n}} \Delta u \Delta v + \vec{\mathbf{r}}''_{vv} \cdot \hat{\mathbf{n}} \Delta v^2\right) + \mathcal{O}(3) \cdot \hat{\mathbf{n}}.$$

The first two linear terms vanish by properties of the triple scalar product. The *second fundamental form* of Σ is defined as

$$\mathrm{I\!I} = L \, \mathrm{d}u^2 + 2M \, \mathrm{d}u \, \mathrm{d}v + N \, \mathrm{d}v^2, \tag{8.1.2}$$

where $L = \vec{\mathbf{r}}''_{uu} \cdot \hat{\mathbf{n}}$, $M = \vec{\mathbf{r}}''_{uv} \cdot \hat{\mathbf{n}}$, and $N = \vec{\mathbf{r}}''_{vv} \cdot \hat{\mathbf{n}}$. Since $\vec{\mathbf{r}}'_u \cdot \hat{\mathbf{n}} = 0$ and $\vec{\mathbf{r}}'_v \cdot \hat{\mathbf{n}} = 0$, by differentiation, we have

$$\begin{gathered}
\vec{\mathbf{r}}''_{uu} \cdot \hat{\mathbf{n}} + \vec{\mathbf{r}}'_u \cdot \hat{\mathbf{n}}'_u = 0, \vec{\mathbf{r}}''_{uv} \cdot \hat{\mathbf{n}} + \vec{\mathbf{r}}'_u \cdot \hat{\mathbf{n}}'_v = 0, \\
\vec{\mathbf{r}}''_{uv} \cdot \hat{\mathbf{n}} + \vec{\mathbf{r}}'_v \cdot \hat{\mathbf{n}}'_u = 0, \vec{\mathbf{r}}''_{vv} \cdot \hat{\mathbf{n}} + \vec{\mathbf{r}}'_v \cdot \hat{\mathbf{n}}'_v = 0.
\end{gathered}$$

It follows that $L = -\vec{\mathbf{r}}'_u \cdot \hat{\mathbf{n}}'_u$, $M = -\vec{\mathbf{r}}'_u \cdot \hat{\mathbf{n}}'_v = -\vec{\mathbf{r}}'_v \cdot \hat{\mathbf{n}}'_u$, and $N = -\vec{\mathbf{r}}'_v \cdot \hat{\mathbf{n}}'_v$. Because $\mathrm{d}\hat{\mathbf{n}} = \hat{\mathbf{n}}'_u \, \mathrm{d}u + \hat{\mathbf{n}}'_v \, \mathrm{d}v$,

$$\mathrm{I\!I} = -\,\mathrm{d}\vec{\mathbf{r}} \cdot \mathrm{d}\hat{\mathbf{n}}.$$

The second fundamental form, in a rough sense, measures the curvature of the surface Σ at P (refer to Figure 19). Both the first and second fundamental forms are geometric invariants; they are independent of the parameterization $\vec{\mathbf{r}}$ of Σ. The first fundamental form is also referred to as the *intrinsic metric* (we will not delve into the metric tensor here) of Σ, and the second fundamental form is an *extrinsic* property of Σ as it is invariant up to the orientation of the surface (consequent direction of the normal vector).

Let $\gamma \subset \Sigma$ be a curve parameterized by arc length, $\vec{\mathbf{r}}(s) = \vec{\mathbf{r}}(u(s), v(s))$. Then the unit tangent vector at $P = \vec{\mathbf{r}}(s)$ is

$$\vec{\mathbf{T}}(s) = \frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}s} = \vec{\mathbf{r}}'_u \frac{\mathrm{d}u}{\mathrm{d}s} + \vec{\mathbf{r}}'_v \frac{\mathrm{d}v}{\mathrm{d}s}.$$

Consequently,

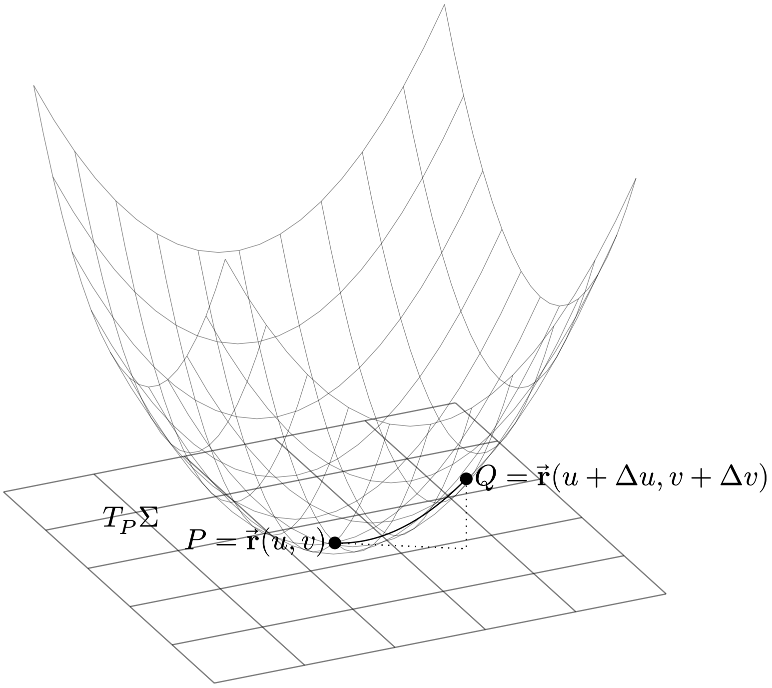

Figure 19: Q has a greater heuristic distance to $T_P\Sigma$ for a more curved surface.

$$\vec{\mathbf{T}}'(s) = \vec{\mathbf{r}}''_{uu}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + 2\vec{\mathbf{r}}''_{uv}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right) + \vec{\mathbf{r}}''_{vv}\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right)^2 + \vec{\mathbf{r}}'_u\frac{\mathrm{d}^2u}{\mathrm{d}s^2} + \vec{\mathbf{r}}'_v\frac{\mathrm{d}^2v}{\mathrm{d}s^2},$$

where the last two terms are in $T_P\Sigma$. Because $\left\|\vec{\mathbf{T}}(s)\right\| = 1$ for all s by the arc-length parameterization, we have

$$0 = \frac{\mathrm{d}\left\|\vec{\mathbf{T}}(s)\right\|^2}{\mathrm{d}s} = \frac{\mathrm{d}\vec{\mathbf{T}}(s)\cdot\vec{\mathbf{T}}(s)}{\mathrm{d}s} = 2\vec{\mathbf{T}}(s)\cdot\vec{\mathbf{T}}'(s).$$

Hence, $\vec{\mathbf{T}}(s)$ and $\vec{\mathbf{T}}'(s)$ are orthogonal and $\vec{\mathbf{T}}'(s)$ is a normal to the curve γ. Let $\hat{\mathbf{n}} = \frac{\vec{\mathbf{r}}'_u\times\vec{\mathbf{r}}'_v}{\|\vec{\mathbf{r}}'_u\times\vec{\mathbf{r}}'_v\|}$ be the unit normal to Σ at P. The *normal curvature* of γ at P in Σ is defined as

$$\kappa_n = \vec{\mathbf{T}}'(s)\cdot\hat{\mathbf{n}} = \left[\vec{\mathbf{r}}''_{uu}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 + 2\vec{\mathbf{r}}''_{uv}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right) + \vec{\mathbf{r}}''_{vv}\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right)^2\right]\cdot\hat{\mathbf{n}}.$$

The quotient

$$\kappa_n = \frac{\mathrm{II}}{\mathrm{I}} = \frac{L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2}{E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2},$$

varies depending on the curve traversing Σ (and ultimately, depending on the direction induced by $\mathrm{d}u$ and $\mathrm{d}v$). On γ, the two representations are equivalent since $\mathrm{I} = \mathrm{d}s^2$. The maximum and minimum values of κ_n are known as the *principal curvatures* κ_1 and κ_2 of Σ at P, achieved along the *principal directions* of the (unit) tangent vectors at P.

The *mean curvature* of Σ at P is defined to be $H = \frac{\kappa_1+\kappa_2}{2}$. Let r_1, r_2 be the radii of curvature corresponding to κ_1 and κ_2. The product of the two principal curvatures is known as the *Gaussian curvature* of Σ at P, denoted by $K = \kappa_1\kappa_2$. We will now heuristically derive the explicit formulas for H and K in terms of E, F, G, L, M, N.

Suppose $p \in \Sigma$. Adopt the matrix notation of $\mathbf{I}$, $\mathbf{II}$ as in

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \mathbf{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix},$$

to reduce to the optimization problem of

$$\kappa_n = \frac{\vec{\mathbf{v}}^\top \mathbf{II} \vec{\mathbf{v}}}{\vec{\mathbf{v}}^\top \mathbf{I} \vec{\mathbf{v}}}, \quad \vec{\mathbf{v}} \in T_p\Sigma.$$

We may restrict $\vec{\mathbf{v}} = (v_1, v_2)$ so that the denominator is always 1, aiming to optimize the numerator. By the method of Lagrange multipliers, we write

$$\mathcal{L}(\vec{\mathbf{v}}, \lambda) = \vec{\mathbf{v}}^\top \mathbf{II} \vec{\mathbf{v}} - \lambda(\vec{\mathbf{v}}^\top \mathbf{I} \vec{\mathbf{v}} - 1).$$

The equation $\boldsymbol{\nabla}\mathcal{L} = \mathbf{0}$ for $\boldsymbol{\nabla} = \left(\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \frac{\partial}{\partial \lambda}\right)$ can then be decomposed into (where $\vec{\mathbf{v}} = (v_1, v_2)$):

$$\begin{gathered} 2Lv_1 + 2Mv_2 - \lambda(2Ev_1 + 2Fv_2) = 0, \\ 2Mv_1 + 2Nv_2 - \lambda(2Fv_1 + 2Gv_2) = 0, \\ (\vec{\mathbf{v}}^\top \mathbf{I} \vec{\mathbf{v}} = 1). \end{gathered}$$

The first two equations can be written as

$$\begin{pmatrix} L - \lambda E & M - \lambda F \\ M - \lambda F & N - \lambda G \end{pmatrix} \vec{\mathbf{v}} = \mathbf{0}. \tag{8.1.3}$$

Let the matrix on the left be denoted by $\mathbf{M}$. In order for non-trivial $(\mathbf{v} \neq \mathbf{0})$ to exist, we must have $\det \mathbf{M} = 0$. That is,

$$\begin{aligned}&(L-\lambda E)(N-\lambda G)-(M-\lambda F)^2\\&\qquad=\lambda^2(EG-F^2)+\lambda(2MF-EN-GL)+LN-M^2=0.\end{aligned}$$

This is a quadratic giving two solutions for λ. From

$$\boldsymbol{\nabla}(\vec{\mathbf{v}}^\top\mathbf{II}\vec{\mathbf{v}})=\lambda\boldsymbol{\nabla}(\vec{\mathbf{v}}^\top\mathbf{I}\vec{\mathbf{v}})$$

it is apparent that the roots $\lambda_1,\lambda_2\in\mathbb{R}$. Moreover, from (8.1.3) we have

$$\mathbf{II}\vec{\mathbf{v}}=\lambda\mathbf{I}\vec{\mathbf{v}}\implies\lambda=\frac{\vec{\mathbf{v}}^\top\mathbf{II}\vec{\mathbf{v}}}{\vec{\mathbf{v}}^\top\mathbf{I}\vec{\mathbf{v}}}.$$

Hence, the two roots λ_1,λ_2 are precisely the principal curvatures. Vieta's formulas give that

$$K=\lambda_1\lambda_2=\frac{LN-M^2}{EG-F^2},\quad H=\frac{EN+GL-2MF}{2EG-2F^2}.$$

Now, assume a parameterization of Σ by $\vec{\mathbf{r}}(u,v)$ (thrice continuously differentiable) such that

$$\mathrm{I}(u,v)=\rho^2\,\mathrm{d}u^2+\rho^2\,\mathrm{d}v^2=\rho^2(\mathrm{d}u^2+\mathrm{d}v^2)$$

(which we will later formalize as a *conformal metric*). Then there is an alternate representation of the Gaussian curvature in terms of ρ.

By definition, $E\equiv G\equiv\rho^2$ while $F\equiv 0$. Moreover,

$$\begin{aligned}LN&=\left(\vec{\mathbf{r}}''_{uu}\cdot\frac{\vec{\mathbf{r}}'_u\times\vec{\mathbf{r}}'_v}{\|\vec{\mathbf{r}}'_u\times\vec{\mathbf{r}}'_v\|}\right)\left(\vec{\mathbf{r}}''_{vv}\cdot\frac{\vec{\mathbf{r}}'_u\times\vec{\mathbf{r}}'_v}{\|\vec{\mathbf{r}}'_u\times\vec{\mathbf{r}}'_v\|}\right)\\&=\frac{\det(\vec{\mathbf{r}}''_{uu}\ \ \vec{\mathbf{r}}'_u\ \ \vec{\mathbf{r}}'_v)\det(\vec{\mathbf{r}}''_{vv}\ \ \vec{\mathbf{r}}'_u\ \ \vec{\mathbf{r}}'_v)}{\|\vec{\mathbf{r}}'_u\|^2\|\vec{\mathbf{r}}'_v\|^2-(\vec{\mathbf{r}}'_u\cdot\vec{\mathbf{r}}'_v)^2}\\&=\frac{\det(\vec{\mathbf{r}}''_{uu}\ \ \vec{\mathbf{r}}'_u\ \ \vec{\mathbf{r}}'_v)\det(\vec{\mathbf{r}}''_{vv}\ \ \vec{\mathbf{r}}'_u\ \ \vec{\mathbf{r}}'_v)}{EG-F^2}\\&=\frac{1}{\rho^4}\det\begin{pmatrix}\vec{\mathbf{r}}''_{vv}\cdot\vec{\mathbf{r}}''_{uu}&\vec{\mathbf{r}}''_{vv}\cdot\vec{\mathbf{r}}'_u&\vec{\mathbf{r}}''_{vv}\cdot\vec{\mathbf{r}}'_v\\\vec{\mathbf{r}}'_u\cdot\vec{\mathbf{r}}''_{uu}&\vec{\mathbf{r}}'_u\cdot\vec{\mathbf{r}}'_u&\vec{\mathbf{r}}'_u\cdot\vec{\mathbf{r}}'_v\\\vec{\mathbf{r}}'_v\cdot\vec{\mathbf{r}}''_{uu}&\vec{\mathbf{r}}'_v\cdot\vec{\mathbf{r}}'_u&\vec{\mathbf{r}}'_v\cdot\vec{\mathbf{r}}'_v\end{pmatrix}.\end{aligned}$$

Similarly,

$$
\begin{aligned}
M^2 &= \frac{1}{\rho^4}\det\begin{pmatrix} \vec{\mathbf{r}}''_{uv}\cdot\vec{\mathbf{r}}''_{uv} & \rho\rho'_v & \rho\rho'_u \\ \rho\rho'_v & \rho^2 & 0 \\ \rho\rho'_u & 0 & \rho^2 \end{pmatrix} \\
&= \frac{1}{\rho^4}\left[\vec{\mathbf{r}}''_{uv}\cdot\vec{\mathbf{r}}''_{uv}\rho^4 - \rho^4(\rho'_v)^2 - \rho^4(\rho'_u)^2\right] \\
&= \vec{\mathbf{r}}''_{uv}\cdot\vec{\mathbf{r}}''_{uv} - (\rho'_v)^2 - (\rho'_u)^2.
\end{aligned}
$$

By differentiation of the equations

$$\vec{\mathbf{r}}'_u\cdot\vec{\mathbf{r}}'_v \equiv F \equiv 0, \quad \vec{\mathbf{r}}'_u\cdot\vec{\mathbf{r}}'_u \equiv E \equiv G \equiv \vec{\mathbf{r}}'_v\cdot\vec{\mathbf{r}}'_v \equiv \rho^2,$$

we have

$$\vec{\mathbf{r}}''_{uu}\cdot\vec{\mathbf{r}}'_v + \vec{\mathbf{r}}'_u\cdot\vec{\mathbf{r}}''_{uv} \equiv 0, \quad \vec{\mathbf{r}}''_{uv}\cdot\vec{\mathbf{r}}'_v + \vec{\mathbf{r}}'_u\cdot\vec{\mathbf{r}}''_{vv} \equiv 0, \tag{8.1.4}$$

and

$$2\vec{\mathbf{r}}''_{uu}\cdot\vec{\mathbf{r}}'_u \equiv 2\rho\rho'_u \equiv 2\vec{\mathbf{r}}''_{uv}\cdot\vec{\mathbf{r}}'_v, \quad 2\vec{\mathbf{r}}''_{uv}\cdot\vec{\mathbf{r}}'_u \equiv 2\rho\rho'_v \equiv 2\vec{\mathbf{r}}''_{vv}\cdot\vec{\mathbf{r}}'_v. \tag{8.1.5}$$

Substituting (8.1.5) into (8.1.4) then gives

$$\vec{\mathbf{r}}''_{uu}\cdot\vec{\mathbf{r}}'_v = -\rho\rho'_v, \quad (\vec{\mathbf{r}}''_{vv}\cdot\vec{\mathbf{r}}'_u = -\rho\rho'_u).$$

Differentiating these give

$$
\begin{gathered}
\vec{\mathbf{r}}'''_{uuv}\cdot\vec{\mathbf{r}}'_v + \vec{\mathbf{r}}''_{uu}\cdot\vec{\mathbf{r}}_{vv} = -(\rho'_v)^2 - \rho\rho''_{vv}, \\
\left(\vec{\mathbf{r}}'''_{vvu}\cdot\vec{\mathbf{r}}'_u + \vec{\mathbf{r}}''_{vv}\cdot\vec{\mathbf{r}}_{uu} = -(\rho'_u)^2 - \rho\rho''_{uu}\right).
\end{gathered}
$$

Differentiating the inner two expressions of (8.1.5), we have

$$
\begin{gathered}
\vec{\mathbf{r}}'''_{uuv}\cdot\vec{\mathbf{r}}'_v + \vec{\mathbf{r}}''_{uv}\cdot\vec{\mathbf{r}}_{uv} = (\rho'_u)^2 + \rho\rho''_{uu}, \\
\left(\vec{\mathbf{r}}'''_{vvu}\cdot\vec{\mathbf{r}}'_u + \vec{\mathbf{r}}''_{uv}\cdot\vec{\mathbf{r}}_{uv} = (\rho'_v)^2 + \rho\rho''_{vv}\right).
\end{gathered}
$$

It follows that

$$\vec{\mathbf{r}}''_{uv}\cdot\vec{\mathbf{r}}_{uv} - \vec{\mathbf{r}}''_{uu}\cdot\vec{\mathbf{r}}''_{vv} = (\rho'_u)^2 + (\rho'_v)^2 + \rho\nabla^2\rho,$$

where ∇^2 here is $\partial^2/\partial u^2 + \partial^2/\partial v^2$. Then

$$
\begin{aligned}
LN &= \frac{1}{\rho^4}\det\begin{pmatrix} \vec{\mathbf{r}}''_{vv}\cdot\vec{\mathbf{r}}''_{uu} & -\rho\rho'_u & \rho\rho'_v \\ \rho\rho'_u & \rho^2 & 0 \\ -\rho\rho'_v & 0 & \rho^2 \end{pmatrix} \\
&= \frac{1}{\rho^4}\left[\vec{\mathbf{r}}''_{vv}\cdot\vec{\mathbf{r}}''_{uu}\rho^4 + \rho^4(\rho'_u)^2 + \rho^4(\rho'_v)^2\right] \\
&= \vec{\mathbf{r}}''_{vv}\cdot\vec{\mathbf{r}}''_{uu} + (\rho'_u)^2 + (\rho'_v)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
M^2 &= \frac{1}{\rho^4} \det \begin{pmatrix} \vec{\mathbf{r}}''_{uv} \cdot \vec{\mathbf{r}}''_{uv} & \rho\rho'_v & \rho\rho'_u \\ \rho\rho'_v & \rho^2 & 0 \\ \rho\rho'_u & 0 & \rho^2 \end{pmatrix} \\
&= \frac{1}{\rho^4} \left[\vec{\mathbf{r}}''_{uv} \cdot \vec{\mathbf{r}}''_{uv} \rho^4 - \rho^4 (\rho'_v)^2 - \rho^4 (\rho'_u)^2 \right] \\
&= \vec{\mathbf{r}}''_{uv} \cdot \vec{\mathbf{r}}''_{uv} - (\rho'_v)^2 - (\rho'_u)^2 .
\end{aligned}
$$

Combining the two expressions, we have

$$
\begin{aligned}
K = \frac{LN - M^2}{EG - F^2} &= \frac{\vec{\mathbf{r}}''_{vv} \cdot \vec{\mathbf{r}}''_{uu} + 2(\rho'_u)^2 + 2(\rho'_v)^2 - \vec{\mathbf{r}}''_{uv} \cdot \vec{\mathbf{r}}''_{uv}}{\rho^4} \\
= \frac{(\rho'_u)^2 + (\rho'_v)^2 - \rho \nabla^2 \rho}{\rho^4} &= -\frac{1}{\rho^2} \nabla^2 (\log \rho).
\end{aligned}
\tag{8.1.6}
$$

To understand the motivation for which $\mathrm{d}s^2$ is said to be conformal, consider two curves in the $u - v$ plane, parameterized by $\gamma_1(t) = (u_1(t), v_1(t))$ and $\gamma_2(t) = (u_2(t), v_2(t))$ such that $\gamma_1(0) = \gamma_2(0) = \mathbf{w}_0 = (u_0, v_0)$. Their images via $\mathbf{r}$ are $\boldsymbol{\alpha}_1(t) = \mathbf{r} \circ \gamma_1(t)$ and $\boldsymbol{\alpha}_2(t) = \mathbf{r} \circ \gamma_2(t)$ so that they intersect at some point $P \in \Sigma$. Let $\gamma'_1(0) = \mathbf{v}_1 = a\mathbf{e}_u + b\mathbf{e}_v$ and $\gamma'_2(0) = \mathbf{v}_2 = c\mathbf{e}_u + d\mathbf{e}_v$ be two tangent vectors. Then the corresponding vectors in $T_P\Sigma$ are

$$
\mathrm{d}\mathbf{r}_{\mathbf{w}_0}(\mathbf{v}_1) = a\, \mathrm{d}\mathbf{r}_{\mathbf{w}_0}(\mathbf{e}_u) + b\, \mathrm{d}\mathbf{r}_{\mathbf{w}_0}(\mathbf{e}_v)
$$

and

$$
\mathrm{d}\mathbf{r}_{\mathbf{w}_0}(\mathbf{v}_2) = c\, \mathrm{d}\mathbf{r}_{\mathbf{w}_0}(\mathbf{e}_u) + d\, \mathrm{d}\mathbf{r}_{\mathbf{w}_0}(\mathbf{e}_v).
$$

Since[1] $\mathbf{e}_u = \frac{\partial}{\partial u}$, $\mathbf{e}_v = \frac{\partial}{\partial v}$ and $\mathrm{d}\mathbf{r}_{\mathbf{w}_0}\big(\frac{\partial}{\partial u}\big) = \frac{\partial \mathbf{r}}{\partial u}\big|_{\mathbf{w}_0}$, $\mathrm{d}\mathbf{r}_{\mathbf{w}_0}\big(\frac{\partial}{\partial v}\big) = \frac{\partial \mathbf{r}}{\partial v}\big|_{\mathbf{w}_0}$. Then

$$
\mathrm{d}\mathbf{r}_{\mathbf{w}_0}(\mathbf{v}_1) = a\mathbf{r}'_u(\mathbf{w}_0) + b\mathbf{r}'_v(\mathbf{w}_0), \quad \mathrm{d}\mathbf{r}_{\mathbf{w}_0}(\mathbf{v}_2) = c\mathbf{r}'_u(\mathbf{w}_0) + d\mathbf{r}'_v(\mathbf{w}_0).
$$

The angle θ_{uv} between $\mathbf{v}_1$ and $\mathbf{v}_2$ on the $u - v$ plane satisfies

$$
\cos \theta_{uv} = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\| \|\mathbf{v}_2\|} = \frac{ac + bd}{\sqrt{a^2 + b^2} \sqrt{c^2 + d^2}},
$$

while the angle θ_Σ between the two tangent vectors in $T_P\Sigma$ satisfies

---

[1] In more modern formulations of differential geometry, differentials (known as pushforwards) are functions mapping tangent vectors to tangent vectors. The notion came from the realization that “changes” of functions are best described in terms of a direction of change. Then basis vectors themselves became partial derivative operators two give the second set of equalities, which is a consequence of the more abstract notion of duality.“ For the remaining sections, this structural viewpoint is not considered.

$$\cos\theta_\Sigma = \left.\frac{(a\mathbf{r}'_u + b\mathbf{r}'_v)\cdot(c\mathbf{r}'_u + d\mathbf{r}'_v)}{\|a\mathbf{r}'_u + b\mathbf{r}'_v\|\|c\mathbf{r}'_u + d\mathbf{r}'_v\|}\right|_{\mathbf{w}_0}$$

$$= \left.\frac{\rho^2(ac+bd)}{\sqrt{a^2\rho^2 + b^2\rho^2}\sqrt{c^2\rho^2 + d^2\rho^2}}\right|_{\mathbf{w}_0} = \cos\theta_{uv}.$$

8.2 Conformal Metrics and Curvature

Let $\Omega \subseteq \mathbb{C}$ be a region and let $\rho \in C^0(\Omega)$ be a positive function. The *conformal metric* (in the following chapters when we refer to *metric* we mean conformal) induced by ρ is given by

$$\mathrm{d}s = \rho(z)|\mathrm{d}z| \quad \text{or} \quad \mathrm{d}s^2 = \rho(z)^2|\mathrm{d}z|^2.$$

The term "conformality" is explained in the previous section (note that this specific usage has little to do with holomorphy). The distance between two points $z_1, z_2 \in \Omega$ is defined as

$$d(z_1, z_2) = \inf_{\gamma\subset\Omega} \int_\gamma \rho(z)|\mathrm{d}z|,$$

where the infimum is taken over all piecewise smooth curves γ in Ω joining z_1 and z_2.

A C^2 metric is said to be *regular*. The (Gaussian) *curvature* of the regular metric ρ at $z \in \Omega$ is defined as

$$K_\rho(z) = -\frac{\nabla^2(\log\rho(z))}{\rho(z)^2}, \tag{8.2.1}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 = 4\partial^2/\partial\bar{z}\partial z$ is the Laplacian operator. This is the same definition as the Gaussian curvature in (8.2.6).

The three following metrics are of particular interest in complex differential geometry:

1 Perhaps the most trivial metric is the *Euclidean metric* (also known as the *parabolic metric*) on $\mathbb{C}$, and is given by

$$\rho = 1, \quad \mathrm{d}s^2 = |\mathrm{d}z|^2.$$

The *Euclidean distance* between two points $z_1, z_2 \in \mathbb{C}$ is

$$\inf_\gamma \int_\gamma |\mathrm{d}z| = |z_2 - z_1|$$

is the length of the straight line segment connecting z_1 and z_2. The group formed by all transformations in the form of $z \mapsto \mathrm{e}^{\mathrm{i}\theta} z + a$ (where $a \in \mathbb{C}$ and $\theta \in \mathbb{R}$) is known as *the group of rigid motions*, or more abstractly, the *special Euclidean group* of order 2, denoted by $\mathrm{SE}(2) < \mathrm{Aut}(\mathbb{C})$, intuitively consists of all rotations and translations and their compositions, while the *Euclidean* group $\mathrm{E}(2) > \mathrm{SE}(2)$ consists of reflections in the form of $z \mapsto \mathrm{e}^{\mathrm{i}\theta} \overline{z} + a$. Obviously, the Euclidean metric is invariant under both groups.

From (8.2.1), we find that Euclidean metric has curvature $K = 0$.

2 The *Poincaré metric* (also referred to as the *hyperbolic metric*) on $\mathbb{D}$ is given by

$$\lambda(z) = \frac{2}{1 - |z|^2}, \quad \mathrm{d}s_\lambda^2 = \frac{4|\mathrm{d}z|^2}{(1 - |z|^2)^2}. \tag{8.2.2}$$

In Lemma 3.5.2, it was shown that the metric is invariant under $\mathrm{Aut}(\mathbb{D})$.

We will now calculate the Poincaré distance between two points $z_1, z_2 \in \mathbb{C}$. First assume the case where $z_1 = 0$ and $z_2 = R \in (0, 1)$. Consider a piecewise smooth curve $\gamma \subset \mathbb{D}$ parameterized by $z(t)$ connecting z_1 and z_2; or in other words

$$z(t) = x(t) + \mathrm{i}y(t), \quad z(0) = z_1 = 0, \quad z(1) = z_2 = R,$$

where $x \in C^1([0, 1])$ and $y \in C^1([0, 1])$ are real-valued functions. Then

$$\begin{aligned}
\int_\gamma \mathrm{d}s &= \int_0^1 \frac{2\sqrt{x'(t)^2 + y'(t)^2}}{1 - x(t)^2 - y(t)^2} \,\mathrm{d}t \\
&\geq \int_0^1 \frac{2|x'(t)|}{1 - x(t)^2} \,\mathrm{d}t \geq \left| \int_0^1 \frac{2x'(t)}{1 - x(t)^2} \,\mathrm{d}t \right| \\
&= \left| \int_0^R \frac{2\,\mathrm{d}x}{1 - x^2} \right| = \log\left(\frac{1 + R}{1 - R}\right).
\end{aligned}$$

Assuming that γ is in the form of $z(t) = Rt, z'(t) = R$ where $t \in [0, 1]$, we have

$$\int_\gamma \mathrm{d}s = \int_0^1 \frac{2R\,\mathrm{d}t}{1 - R^2 t^2} = \log\left(\frac{1 + R}{1 - R}\right).$$

Hence, the Poincaré distance between 0 and R is given by

$$d(0, R) = \log\left(\frac{1 + R}{1 - R}\right)$$

and the straight line segment connecting the two points is a *geodesic* (path of least length under a metric or other criteria). For fixed $\theta \in \mathbb{R}$ since $z \mapsto z\mathrm{e}^{\mathrm{i}\theta} \in \operatorname{Aut}(\mathbb{D})$, by the Schwarz–Pick Lemma (Lemma 3.5.2), we have

$$d(0,R) = d\big(0, R\mathrm{e}^{\mathrm{i}\theta}\big) = \log\left(\frac{1+R}{1-R}\right)$$

by the invariance under $\operatorname{Aut}(\mathbb{D})$. Now let z_1 and z_2 be arbitrary points in $\mathbb{D}$. The Möbius transformation

$$\varphi_{z_1}(z) = \frac{z - z_1}{1 - \overline{z_1} z}$$

maps z_1 to 0 and maps z_2 to $\frac{z_2 - z_1}{1-\overline{z_1}z_2}$. Hence, we have

$$d(z_1, z_2) = d\left(0, \frac{z_2 - z_1}{1 - \overline{z_1} z_2}\right) = \log\left[\frac{1 + \left|\frac{z_2 - z_1}{1-\overline{z_1}z_2}\right|}{1 - \left|\frac{z_2 - z_1}{1-\overline{z_1}z_2}\right|}\right] = \inf_{\gamma} \int_{\gamma} \mathrm{d}s,$$

which is the Poincaré distance (or *hyperbolic distance*) between z_1 and z_2. The infimum is attained along the geodesic curve γ parameterized by

$$z(t) = \varphi_{z_1}^{-1}\left(\frac{z_2 - z_1}{1 - \overline{z_1} z_2} t\right)$$

for $t \in [0,1]$. By Theorem 5.1.4, the geodesic is either an arc or a straight line segment passing through z_1 and z_2. Since $\partial\mathbb{D}$ is orthogonal to the straight line passing through 0 and $\frac{z_2 - z_1}{1-\overline{z_1}z_2}$, by the conformality of $\varphi_{z_1}^{-1}$, $\varphi_{z_1}^{-1}(\partial\mathbb{D}) = \partial\mathbb{D}$ is orthogonal to the circular (or straight line) extension of the geodesic curve.

As a consequence of the Schwarz–Pick Lemma (Lemma 3.5.2), for any $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, we have

$$d(f(z_1), f(z_2)) \leq d(z_1, z_2),$$

where equality is attained iff $f \in \operatorname{Aut}(\mathbb{D})$. The Poincaré metric has constant negative curvature -1 since

$$\begin{aligned} K_\lambda &= -\frac{4}{\lambda^2} \frac{\partial^2 \log \circ \lambda}{\partial \overline{z} \partial z} = -\frac{4}{\lambda^2} \frac{\partial}{\partial \overline{z}} \left(\frac{\lambda'_z}{\lambda}\right) = -\frac{2}{\lambda^2} \frac{\partial}{\partial \overline{z}} \left(\frac{2\overline{z}}{1 - |z|^2}\right) \\ &= -\frac{2}{\lambda^2}(\lambda + \overline{z}\lambda'_{\overline{z}}) = -\frac{(1-|z|^2)^2}{2}\left(\lambda + \overline{z}\frac{2z}{(1-|z|^2)^2}\right) \\ &= -(1 - |z|^2) - |z|^2 = -1, \end{aligned}$$

where $\lambda'_z = \partial\lambda/\partial z$ and $\lambda'_{\overline{z}} = \partial\lambda/\partial\overline{z}$.

3 The *spherical metric* (also referred to as the *elliptic metric*) on $\hat{\mathbb{C}}$ is given by

$$\sigma(z) = \frac{2}{1+|z|^2}, \quad \mathrm{d}s_\sigma^2 = \frac{4|\mathrm{d}z|^2}{\left(1+|z|^2\right)^2}. \tag{8.2.3}$$

Under the inverse stereographic projection of $S^2 \to \hat{\mathbb{C}}$, for a given $z \in \hat{\mathbb{C}}$, the corresponding point in S^2 is

$$(x_1, x_2, x_3) = \left(\frac{z+\overline{z}}{|z|^2+1}, \frac{z-\overline{z}}{\mathrm{i}|z|^2+\mathrm{i}}, \frac{|z|^2-1}{|z|^2+1} \right).$$

If we let $P = (x_1, x_2, x_3)$ and $Q = (\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3})$ be two points in S^2, the distance between the two points is the length of the shortest arc $\widehat{x}$ (a subset of great circle passing the two points). By considering P and Q as vectors from $(0,0,0)$, this distance is equal to

$$\begin{aligned}
\arccos(P \cdot Q) &= 2\arctan\sqrt{\frac{1 - x_1\widetilde{x_1} - x_2\widetilde{x_2} - x_3\widetilde{x_3}}{1 + x_1\widetilde{x_1} + x_2\widetilde{x_2} + x_3\widetilde{x_3}}} \\
&= 2\arctan\sqrt{\frac{1 - \frac{(z+\overline{z})(\tilde{z}+\overline{\tilde{z}})}{(|z|^2+1)(|\tilde{z}|^2+1)} + \frac{(z-\overline{z})(\tilde{z}-\overline{\tilde{z}})}{(|z|^2+1)(|\tilde{z}|^2+1)} - \frac{(|z|^2-1)(|\tilde{z}|^2-1)}{(|z|^2+1)(|\tilde{z}|^2+1)}}{1 + \frac{(z+\overline{z})(\tilde{z}+\overline{\tilde{z}})}{(|z|^2+1)(|\tilde{z}|^2+1)} - \frac{(z-\overline{z})(\tilde{z}-\overline{\tilde{z}})}{(|z|^2+1)(|\tilde{z}|^2+1)} + \frac{(|z|^2-1)(|\tilde{z}|^2-1)}{(|z|^2+1)(|\tilde{z}|^2+1)}}} \\
&= 2\arctan\sqrt{\frac{-z\overline{\tilde{z}} - \overline{z}\tilde{z} + |z|^2 + |\tilde{z}|^2}{z\overline{\tilde{z}} + \overline{z}\tilde{z} + |z|^2|\tilde{z}|^2 + 1}} = 2\arctan\sqrt{\frac{(z-\tilde{z})\left(\overline{z}-\overline{\tilde{z}}\right)}{\left(z\overline{\tilde{z}}+1\right)(\overline{z}\tilde{z}+1)}}.
\end{aligned}$$

Notice that the fraction within the square root is a product between a complex number and its conjugate. Thus, this distance is equal to

$$d(z, \tilde{z}) = 2\arctan\left|\frac{z-\tilde{z}}{z\overline{\tilde{z}}+1}\right|$$

in the extended complex plane. Let $\tilde{z} = z + \Delta z$. It follows that

$$
\begin{aligned}
d(z, z+\Delta z) &= 2\arctan\left|\frac{\Delta z}{|z|^2 + z\overline{\Delta z} + 1}\right| \\
&= 2\arctan\left|\frac{\Delta z}{|z|^2+1}\frac{1}{1+\frac{z\overline{\Delta z}}{|z|^2+1}}\right| \\
&= 2\arctan\left|\frac{\Delta z}{|z|^2+1}(1+\mathcal{O}(\Delta z))\right| = 2\arctan\left|\frac{\Delta z}{|z|^2+1} + \mathcal{O}(\Delta z^2)\right| \\
&= 2\left[\left|\frac{\Delta z}{|z|^2+1} + \mathcal{O}(\Delta z^2)\right| + \mathcal{O}\left(\Delta z^3\left[\frac{1}{|z|^2+1} + \mathcal{O}(\Delta z)\right]^3\right)\right] \\
&= 2\left[\left|\frac{\Delta z}{|z|^2+1} + \mathcal{o}(\Delta z^2)\right|\right],
\end{aligned}
$$

where we have taken the liberty to coalesce orders for simplification. Since

$$
\lim_{\Delta z \to 0}\left|\frac{d(z, z+\Delta z)}{\Delta z}\right| = \frac{2}{|z|^2+1},
$$

the metric as defined in (8.2.3) has a clear geometric meaning: the distance between two points z and $\tilde{z}$ under the metric in (8.2.3) is the shortest distance between the corresponding points in S^2, or their spherical distance.

Thus, if curve γ joins z and $\tilde{z}$, we have

$$
d(z, \tilde{z}) = \inf_{\gamma}\int_{\gamma}\sigma(z)|\mathrm{d}z|,
$$

which attains its infimum when the inverse stereographic projection of γ is a great circle of S^2. Thus, σ is known as the spherical metric.

The corresponding curvature is given by

$$
\begin{aligned}
K_\sigma &= -\frac{4}{\sigma^2}\frac{\partial^2}{\partial\overline{z}\partial z}(\log\sigma(z)) = -\frac{4}{\sigma^2}\frac{\partial}{\partial\overline{z}}\left(\frac{\sigma'_z}{\sigma}\right) = \frac{2}{\sigma^2}\frac{\partial}{\partial\overline{z}}\left(\frac{2\overline{z}}{1+|z|^2}\right) \\
&= \frac{2}{\sigma^2}(\sigma + \overline{z}\sigma'_{\overline{z}}) = \frac{\left(1+|z|^2\right)^2}{2}\left(\frac{2}{1+|z|^2} - \frac{2|z|^2}{\left(1+|z|^2\right)^2}\right) \\
&= (1+|z|^2) - |z|^2 = 1,
\end{aligned}
$$

where $\sigma'_z = \partial\sigma/\partial z$ and $\sigma'_{\overline{z}} = \partial\sigma/\partial\overline{z}$. This can also be verified by computing the principal curvatures of the unit sphere, which are both one.

The importance of the selected regions lies in the uniformization to be mentioned in @ sec:riemannsurfaces.

Let Ω_1 and Ω_2 be two open regions in $\mathbb{C}$ such that $f : \Omega_1 \to \Omega_2$ is univalent (implying that $f' \neq 0$ by Lemma 5.1.1). If ρ is a metric on Ω_2, then

$$f^*\rho = (\rho \circ f)|f'| \tag{8.2.4}$$

defines a metric on Ω_1, referred to as the *metric pullback of ρ by f*.

Curvature as defined in (8.2.1) is invariant under pullbacks of conformal mappings, or in the case above, we now aim to show that (under assumptions of regularity)

$$K_\rho(f(z)) = K_{f^*\rho}(z). \tag{8.2.5}$$

By explicit definition,

$$K_{f^*\rho}(z) = -\frac{\nabla^2(\log \circ f^*\rho(z))}{(f^*\rho)(z)^2} = -\frac{\nabla^2(\log \circ \rho \circ f)(z) + \nabla^2(\log|f'(z)|)}{(f^*\rho)(z)^2}.$$

Since $f'(z) \neq 0$, $\log \circ |f'| = \mathfrak{Re}\log(f')$ is harmonic on Ω_1 with a vanishing Laplacian. Hence,

$$\begin{aligned}
K_{f^*\rho}(z) &= -\frac{\nabla^2(\log \circ \rho \circ f(z))}{(\rho \circ f)^2|f'|^2} = -\frac{4}{(\rho \circ f)^2|f'|^2}\frac{\partial}{\partial \overline{z}}\left(\frac{\partial}{\partial z}(\log \circ \rho \circ f(z))\right) \\
&= -\frac{4}{(\rho \circ f)^2|f'|^2}\frac{\partial}{\partial \overline{z}}\left(\frac{\partial}{\partial f}(\log \circ \rho \circ f)\frac{\partial f}{\partial z} + \frac{\partial}{\partial \overline{f}}(\log \circ \rho \circ f)\frac{\partial \overline{f}}{\partial z}\right) \\
&= -\frac{4}{(\rho \circ f)^2|f'|^2}\frac{\partial}{\partial \overline{z}}\left(\frac{\partial \log \circ \rho \circ f}{\partial f}\frac{\partial f}{\partial z} + \frac{\partial \log \circ \rho \circ f}{\partial \overline{f}}\overline{\left(\frac{\partial f}{\partial \overline{z}}\right)}\right) \\
&= -\frac{4}{(\rho \circ f)^2|f'|^2}\frac{\partial f}{\partial z}\frac{\partial}{\partial \overline{z}}\left(\frac{\partial \log \circ \rho \circ f}{\partial f}\right) \\
&= -\frac{4}{(\rho \circ f)^2|f'|^2}\frac{\partial f}{\partial z}\left(\frac{\partial^2 \log \circ \rho \circ f}{\partial f^2}\frac{\partial f}{\partial \overline{z}} + \frac{\partial^2 \log \circ \rho \circ f}{\partial \overline{f}\partial f}\overline{\left(\frac{\partial f}{\partial z}\right)}\right) \\
&= -\frac{4}{(\rho \circ f)^2}\frac{\partial^2 \log \circ \rho \circ f}{\partial \overline{f}\partial f} = -\frac{\nabla_f^2(\log \circ \rho \circ f)}{(\rho \circ f)^2} = K_\rho(f(z)).
\end{aligned}$$

For a given metric $\mathrm{d}s = \lambda(z)|\mathrm{d}z|$, if there is some other parameterization such that $\mathrm{d}s = \lambda'(z')|\mathrm{d}z'|$, $z' = f(z)$ is conformal, then the relation is given by $\lambda = f^*\lambda'$. Under differing parameterizations of a metric $\mathrm{d}s$, we once again have the invariance of curvature.

8.3 From Schwarz–Pick to Ahlfors and Value Distribution of Entire Functions

While Schwarz Lemma in Lemma 3.5.1 concerns self-maps of $\mathbb{D}$ with a fixed point at the origin, the Schwarz–Pick Lemma in Lemma 3.5.2 generalizes this to arbitrary points in $\mathbb{D}$ as well as the hyperbolic contraction property of holomorphic maps.

In 1938, Lars Ahlfors provided a further generalization by curvature, prompting the study of complex functions from a differential-geometric approach.

The hyperbolic metric λ in (8.3.2) does not increase under any holomorphic $f : \mathbb{D} \to \mathbb{D}$. It was realized that this was a consequence of the constant negative curvature -1 of λ. The results we now provide are simplifications of those from [2].

Theorem 8.3.1 *(Schwarz–Ahlfors–Pick)*: Let f be holomorphic on $\mathbb{D}$. Suppose that ρ is a regular metric defined on an open neighborhood U, where $f(\mathbb{D}) \subseteq U$, $\mathrm{d}s_\rho^2 = \rho^2(w)|\mathrm{d}w|^2$, and $K_\rho(w) \leq -1$ for all $w \in U$. Then

$$f^*\rho(z) \leq \lambda(z) \quad \forall z \in \mathbb{D},$$

where λ is the Poincaré metric, and equivalently,

$$\mathrm{d}s_{f^*\rho}^2 \leq \mathrm{d}s_\lambda^2,$$

or that the metric ρ does not exceed the hyperbolic metric under the map f.

Proof: Define

$$\lambda_r(z) = \left(z \mapsto \frac{z}{r}\right)^* \lambda(z) = \frac{2r}{r^2 - |z|^2}, \quad 0 < r < 1 \tag{8.3.1}$$

to generalize the Poincaré metric to $D(0,r)$. (8.3.5) gives that $K_{\lambda_r}(z) = K_\lambda\left(\frac{z}{r}\right) = -1$ for any $z \in D(0,r)$. Define the real-valued function

$$u_r(z) = \frac{f^*\rho(z)}{\lambda_r(z)} \quad \text{for} \quad z \in D(0,r),$$

which is nonnegative and continuous on $D(0,r)$. The pullback metric $f^*\rho = (\rho \circ f)|f'|$ is continuous on $\mathbb{D}$ and thus bounded on $\overline{D(0,r)}$ (as a consequence of Theorem 1.2.13). As $|z| \to r^-$, $\lambda_r(z) \to \infty$, and hence $\lim_{|z|\to r^-} u_r(z) = 0$. Thus,

$$M_r = \max_{z \in \overline{D(0,r)}} u_r(z)$$

must be attained at some $z = \tau_r \in D(0,r)$ (within the interior).

If $M_r = 0$, then $\forall z \in D(0,r)$, $\frac{f^*\rho(z)}{\lambda_r(z)} = 0 \Longrightarrow M_r = 0 \leq 1$ by maximality. On the contrary, if $M_r > 0$, $f^*\rho$ has well-defined Gaussian curvature at τ_r. Since

$$\begin{aligned}(\nabla^2 \log(u_r))(\tau_r) &= (\nabla^2 \log(f^*\rho))(\tau_r) - (\nabla^2 \log(\lambda_r))(\tau_r) \\ &= -K_{f^*\rho}(\tau_r) f^*\rho(\tau_r)^2 + K_{\lambda_r}(\tau_r)\lambda_r(\tau_r)^2 \\ &= -K_{f^*\rho}(\tau_r) f^*\rho(\tau_r)^2 - \lambda_r(\tau_r)^2.\end{aligned} \tag{8.3.2}$$

By assumption, we have $-K_{f^*\rho}(\tau_r) \geq 1$. Hence, $(\nabla^2 \log(u_r))(\tau_r) \geq f^*\rho(\tau_r)^2 - \lambda_r(\tau_r)^2$. Since log is increasing in $\mathbb{R}$, τ_r is a local maximum of $\log \circ u_r$ and hence $(\nabla^2 \log(u_r))(\tau_r) \leq 0$. Thus, we have

$$f^*\rho(\tau_r)^2 - \lambda_r(\tau_r)^2 \leq 0 \Longleftrightarrow M_r \leq 1.$$

Now let $r \to 1^-$, and it follows that $M_r \to \sup_{z\in\mathbb{D}} \frac{f^*\rho(z)}{\lambda(z)} \leq 1$. □

Theorem 8.3.2: Let $f : \mathbb{D} \to U \subseteq \mathbb{C}$ be holomorphic. Let $\mathrm{d}s = \rho(w)|\mathrm{d}w|$ (where $\rho : U \to \mathbb{R}_{>0}$) define a regular metric such that at every point $w \in U$, either

1 The second derivatives of $\log \lambda$ are continuous (C^2) and

$$\nabla^2(\log \lambda)(w) \geq \lambda^2$$

2 There exist two opposite directions $\hat{\mathbf{n}}'$, $\hat{\mathbf{n}}''$ such that

$$\nabla_{\hat{\mathbf{n}}'}(\log \rho)(w) + \nabla_{\hat{\mathbf{n}}''}(\log \rho)(w) > 0 \tag{8.3.3}$$

(the directional derivatives).

Then the metric $f^*\rho$ does not exceed the hyperbolic metric λ.

Proof: The first case is equivalent to $K_\rho \leq -1$.

The only modification to the proof of Theorem 8.3.1 is to consider the case of the inequality involving directional derivatives for each τ_r (which by definition, is where the maximum value of u_r is attained within $D(0,r)$). By the increasing nature of log, τ_r is also a local maximum of $\phi \equiv \log u_r \equiv \log(f^*\rho) - \log \lambda_r$.

Since τ_r is a local maximum, we must have

$$\nabla_{\hat{\mathbf{n}}'}\phi(\tau_r) \leq 0, \quad \nabla_{\hat{\mathbf{n}}''}\phi(\tau_r) \leq 0 \Longrightarrow \nabla_{\hat{\mathbf{n}}'}\phi(\tau_r) + \nabla_{\hat{\mathbf{n}}''}\phi(\tau_r) \leq 0.$$

This implies that

$$\begin{aligned}&\nabla_{\hat{\mathbf{n}}'}(\log f^*\rho)(\tau_r) + \nabla_{\hat{\mathbf{n}}''}(\log f^*\rho)(\tau_r) \\ &\qquad \leq \nabla_{\hat{\mathbf{n}}'}(\log \lambda_r)(\tau_r) + \nabla_{\hat{\mathbf{n}}''}(\log \lambda_r)(\tau_r) = 0\end{aligned}$$

by the symmetry of the hyperbolic metric and the fact that the two directions are opposite to each other. Pulling back to ρ contradicts with (8.3.3). Thus, τ_r cannot both simultaneously be the location of a maximum while satisfying said inequality; therefore the theorem follows. ☐

Theorem 8.3.3: Let $f : \mathbb{D} \to U$ be holomorphic. Let

$$\rho : U \subseteq \mathbb{C} \to \mathbb{R}_{>0}, \quad \mathrm{d}s_\rho = \rho(w)|\mathrm{d}w|$$

be a continuous conformal metric (but not necessarily C^2) such that at each point w, there exists a neighborhood $V_w \ni w$ in U and a regular metric ρ_w thereon such that $\rho_w(w) = \rho(w)$ and $\rho_w \leq \rho$ everywhere else (referred to as a “supporting metric”). If each $K_{\rho_w} \leq -1$ everywhere, then the conclusion of Theorem 8.3.1 continues to hold for ρ.

Proof: By assumption, we have

$$f^* \rho_{\tau_r}(\tau_r) = f^* \rho(\tau_r), \quad f^* \rho_{\tau_r}(z) \leq f^* \rho(z) \quad \text{for} \quad z \neq \tau_r.$$

Let $\tilde{u}_r(z) = \frac{f^* \rho_{\tau_r}(z)}{\lambda_r(z)}$, which attains is maximum of M_r at τ_r as well.

The calculations in Theorem 8.3.1 on $\tilde{u}_r(z)$ (whose curvature calculations are now valid by C^2) give that $f^* \rho_{\tau_r}(\tau_r)^2 - \lambda_r(\tau_r)^2 \leq 0$, which implies $M_r \leq 1$. (We have used the supporting metric, rather than ρ, to derive this inequality) The rest of the theorem follows naturally. ☐

Theorem 8.3.1 generalizes the Schwarz–Pick Theorem when ρ is chosen to be λ and f is chosen such that $f(\mathbb{D}) \subseteq \mathbb{D}$.

For the purpose of the proceeding generalization, we define the conformal metric

$$\lambda_r^\alpha(z) = \frac{1}{\sqrt{\alpha}} \left(z \mapsto \frac{z}{r} \right)^* \lambda(z) = \frac{2r}{\sqrt{\alpha}(r^2 - |z|^2)}, \quad r > 0, z \in D(0, r) \tag{8.3.4}$$

Its Gaussian curvature is

$$\begin{aligned} K_{\lambda_r^\alpha}(z) &= -4 \frac{\partial^2}{\partial \bar{z} \partial z} \left[\log \left(\frac{2r}{\sqrt{\alpha}(r^2 - |z|^2)} \right) \right] \left(\frac{\sqrt{\alpha}(r^2 - |z|^2)}{2r} \right)^2 \\ &= -4\alpha \frac{\partial^2}{\partial \bar{z} \partial z} \left[\log \left(\frac{2r}{r^2 - |z|^2} \right) \right] \left(\frac{r^2 - |z|^2}{2r} \right)^2 = \alpha K_{\lambda_r}(z) = -\alpha, \end{aligned}$$

via the results and definitions in (8.3.1).

Corollary 8.3.3.1: Let $r > 0$ and suppose $f : D(0, r) \to U$ is holomorphic, where $U \subseteq \mathbb{C}$ is a region. For any $\beta > 0$, define ρ to be a regular metric on U with $\mathrm{d}s_\rho^2 = \rho^2(w)|\mathrm{d}w|^2$ such that

$$K_\rho(w) \leq -\beta, \quad \forall w \in U.$$

Then $\forall \alpha > 0$,

$$f^* \rho(z) \leq \sqrt{\frac{\alpha}{\beta}} \lambda_r^\alpha(z)$$

for any $z \in D(0, r)$, where $f^* \rho(z) = (\rho \circ f)|f'|$ is the metric pullback.

Proof: Consider the $(z \mapsto zr)^* f^* \big(\rho\sqrt{\beta}\big)$, a conformal metric pullback of $\rho\sqrt{\beta}$ to $\mathbb{D}$, which satisfies

$$K_{(z \mapsto zr)^* f^* (\rho\sqrt{\beta})} \leq -1.$$

By Schwarz–Ahlfors–Pick (Theorem 8.3.1), we have

$$(z \mapsto zr)^* f^* \Big(\rho\sqrt{\beta}\Big)(z) \leq \lambda(z) = \sqrt{\alpha}((z \mapsto rz)^* \lambda_r^\alpha)(z) \quad \text{for} \quad z \in \mathbb{D}.$$

Since $r \neq 0$, this implies that

$$f^* \Big(\rho\sqrt{\beta}\Big)(z) \leq \sqrt{\alpha} \lambda_r^\alpha(z) \quad \text{for} \quad z \in D(0, r).$$

Since $\sqrt{\beta}$ is a constant,

$$\sqrt{\beta} f^* \rho(z) \leq \sqrt{\alpha} \lambda_r^\alpha(z), \quad \forall z \in D(0, r). \qquad \square$$

Corollary 8.3.3.2 (*Generalized Liouville*): If $f : \mathbb{C} \to U$ is entire and U admits a regular metric of curvature bounded above by a negative constant, then f must be constant.

Proof: By assumption, $\exists \beta > 0$ such that $\sup_{w \in U} K_\rho(w) \leq -\beta$. Then Corollary 8.3.3.1 gives that

$$f^* \rho(z) \leq \frac{1}{\sqrt{\beta}} \lambda_r(z) \quad \forall z \in D(0, r)$$

for any $r > 0$. As $r \to \infty$, $\lambda_r \to 0$. Hence, $f^* \rho(z) = 0$, implying that $(\rho \circ f)(z)|f'| = 0$. Hence, f is constant. $\square$

Remark: Corollary 8.3.3.2 implies Liouville's Theorem (Theorem 3.2.3). To justify this differential-geometric generalization, suppose $f : \mathbb{C} \to U$ is entire such that U is bounded. There then exists some $R > 0$ such that $U \subseteq D(0, R)$.

The metric λ_R has constant negative curvature $K = -1$ on $D(0, R)$, and hence, under $\beta = 1$, Corollary 8.3.3.2 implies that f is constant.

It is understood that an entire function is guaranteed to be constant if it is bounded. This is a statement of sufficiency, but it begs the question of the capacity for possible generalization of boundedness under which constancy is still always satisfied.

Consider an entire function $f : \mathbb{C} \to U$, where U is an unbounded region such that $\mathbb{C} \setminus U$ has positive area. Fix $\zeta \in \operatorname{int}(\mathbb{C} \setminus U)$. Then the map $z \mapsto \frac{1}{z-\zeta}$ maps U to a bounded region and hence $z \mapsto \frac{1}{f(z)-\zeta}$ is constant by Liouville's Theorem (Theorem 3.2.3), implying the constancy of f (the essential proof of Theorem 4.2.4).

In contrast, if $f : \mathbb{C} \to U$ is entire and $\mathbb{C} \setminus U$ has zero area (one readily considers sets consisting of curves or isolated points), we must be more specific in determining sufficient conditions that still imply constancy of f.

Similar to in the proof of the Riemann Mapping Theorem (Theorem 5.3.1), one may use holomorphic square roots or other transformations to reduce to the bounded setting.

Example 8.3.1: If $f : \mathbb{C} \to \mathbb{C} \setminus \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ is entire, then f must be constant.

Proof: Consider the biholomorphism $\varphi(z) = \frac{1}{z}$, mapping $\mathbb{C} \setminus \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ to $\mathbb{C}^* \setminus \mathbb{R}_{\geq 1}$. By simple connectivity of $\mathbb{C} \setminus \mathbb{R}_{\geq 1}$, there exists a univalent branch ψ of $z \mapsto \sqrt{z-1}$ on $\mathbb{C} \setminus \mathbb{R}_{\geq 1}$. Now omitting the origin, it is trivially realized that $\psi(\mathbb{C}^* \setminus \mathbb{R}_{\geq 1}) \cap -\psi(\mathbb{C}^* \setminus \mathbb{R}_{\geq 1}) = \varnothing$. If otherwise, then $\exists \xi \in \psi(\mathbb{C}^* \setminus \mathbb{R}_{\geq 1})$ such that $-\xi \in \psi(\mathbb{C}^* \setminus \mathbb{R}_{\geq 1})$, implying that $\exists z_1, z_2 \in \mathbb{C}^* \setminus \mathbb{R}_{\geq 1}$ such that $\phi(z_1) = \xi$ and $\phi(z_2) = -\xi$, implying that $z_1 = z_2$ and $\xi = 0 \Longrightarrow z_1 = z_2 = 1$, which does not lie in $\psi(\mathbb{C}^* \setminus \mathbb{R}_{\geq 1})$.

Now fix $\xi \in \psi(\mathbb{C}^* \setminus \mathbb{R}_{\geq 1})$. By the Open Mapping Theorem (Theorem 5.1.1), $\exists \varepsilon > 0$ such that $D(\xi, \varepsilon) \subseteq \psi(\mathbb{C}^* \setminus \mathbb{R}_{\geq 1})$. Consequently, $D(-\xi, \varepsilon) \cap \psi(\mathbb{C}^* \setminus \mathbb{R}_{\geq 1}) = \varnothing$. Lastly, the function $\phi(z) = \frac{\varepsilon}{z+\xi}$ maps $\psi(\mathbb{C}^* \setminus \mathbb{R}_{\geq 1})$ to $\mathbb{D}$. By Liouville (Theorem 3.2.3), $\phi \circ \psi \circ \varphi \circ f$ is constant, which implies f is constant by the injectivity of ϕ, ψ, and φ. $\square$

The preceding examples show that if the omitted set is sufficiently "large" (in the sense of having positive area or disconnecting the plane in certain ways), then any entire function avoiding it must reduce to a constant. However, there are natural limits to the smallness of the omitted set. For instance, the exponential function exp is an entire non-constant function whose image is $\mathbb{C}^*$, omitting only a single point. Thus, the property that *an entire function* *omits a set* is not by itself sufficient to guarantee constancy unless that set is suitably substantial. This observation is formalized by Picard's Little Theorem (Theorem 8.3.4), which as preluded to before, asserts that any non-constant entire function can omit at most one complex value.

Proposition 8.3.1: Let $U \subset \mathbb{C}$ be an open set such that $\mathbb{C} \setminus U$ contains at least two points. Then U admits a conformal metric $\rho \in C^2(U)$, $\mathrm{d}s_\rho^2 = \rho^2(z)|\mathrm{d}z|^2$ such that

$$K_\rho(z) \leq -\beta < 0 \quad \forall z \in U$$

for some $\beta > 0$.

Proof: Without loss of generality, we may assume that $\{0,1\} \subseteq \mathbb{C} \setminus U$ (if not, a linear transformation $z \mapsto \frac{z-\xi_1}{\xi_2-\xi_1}$ where $\xi_1, \xi_2 \in \mathbb{C} \setminus U$ are distinct will suffice to transform U to such a region).

Define a regular metric with

$$\rho(z) = \frac{\sqrt{1+|z|^{\frac{1}{3}}}\sqrt{1+|z-1|^{\frac{1}{3}}}}{|z|^{\frac{5}{6}}|z-1|^{\frac{5}{6}}}, \quad \mathrm{d}s_\rho^2 = \rho^2(z)|\mathrm{d}z|^2 \tag{8.3.5}$$

on $\mathbb{C} \setminus \{0,1\}$.

Since $\nabla^2\left(\log|z|^{\frac{5}{6}}\right) = \frac{5}{6}\nabla^2(\log|z|) = \frac{5}{6}\nabla^2(\mathfrak{Re}\log(z))$,

$$\begin{aligned}\nabla^2 \log\left(\frac{\sqrt{1+|z|^{\frac{1}{3}}}}{|z|^{\frac{5}{6}}}\right) &= 2\frac{\partial^2}{\partial\overline{z}\partial z}\left(\log\left(1+|z|^{\frac{1}{3}}\right)\right) = \frac{z^{-\frac{5}{6}}}{3}\frac{\partial}{\partial\overline{z}}\left(\frac{\overline{z}^{\frac{1}{6}}}{1+|z|^{\frac{1}{3}}}\right) \\ &= \frac{z^{-\frac{5}{6}}}{3}\frac{\partial}{\partial\overline{z}}\left(\frac{\overline{z}^{\frac{1}{6}}}{1+|z|^{\frac{1}{3}}}\right) = \frac{z^{-\frac{5}{6}}\overline{z}^{-\frac{5}{6}}\left(1+|z|^{\frac{1}{3}}\right) - z^{-\frac{5}{6}}\overline{z}^{\frac{1}{6}}z^{\frac{1}{6}}\overline{z}^{-\frac{5}{6}}}{18\left(1+|z|^{\frac{1}{3}}\right)^2} \\ &= \frac{1}{18|z|^{\frac{5}{3}}\left(1+|z|^{\frac{1}{3}}\right)^2},\end{aligned}$$

and a similar calculation yields

$$\nabla^2 \log\left(\frac{\sqrt{1+|z-1|^{\frac{1}{3}}}}{|z-1|^{\frac{5}{6}}}\right) = \frac{1}{18|z-1|^{\frac{5}{3}}\left(1+|z-1|^{\frac{1}{3}}\right)^2}.$$

Hence,

$$K_\rho(z) = -\frac{1}{18}\left[\frac{|z-1|^{\frac{5}{3}}}{\left(1+|z|^{\frac{1}{3}}\right)^3\left(1+|z-1|^{\frac{1}{3}}\right)} + \frac{|z|^{\frac{5}{3}}}{\left(1+|z-1|^{\frac{1}{3}}\right)^3\left(1+|z|^{\frac{1}{3}}\right)}\right],$$

and that

1 $K_\rho \in C^0(\mathbb{C} \setminus \{0,1\})$.
2 $\forall z \in \mathbb{C} \setminus \{0,1\}$, $K_\rho(z) < 0$.
3 $\lim_{z\to 0} K_\rho(z) = -\frac{1}{36}$.
4 $\lim_{z\to 1} K_\rho(z) = -\frac{1}{36}$.
5 $\lim_{z\to\infty} K_\rho(z) = -\infty$ in any direction (as in the one-point compactification).

Hence, $\exists\delta > 0$ such that $\left|K_\rho(z) + \frac{1}{36}\right| < \frac{1}{72}$ for any $z \in D^*(0,\delta) \cup D^*(1,\delta)$ and $\exists R > 0$ such that $K_\rho(z) < -1$ for any z satisfying $|z| > R$. By compactness of $\overline{D(0,R)} \setminus (D(0,\delta) \cup D(1,\delta))$ and continuity, it attains its supremum of some value $-M < 0$ by Theorem 1.2.14. Let $-\beta = \max\left\{-\frac{1}{72}, -M\right\} < 0$.

$$\therefore \quad K_\rho(z) \le -\beta < 0 \quad \forall z \in \mathbb{C} \setminus \{0,1\}. \qquad \square$$

And we have the final implication:

Theorem 8.3.4 *(Picard's Little Theorem)*: Let $f : \mathbb{C} \to U$ be entire such that $\mathbb{C} \setminus U$ contains two or more points. Then f is constant.

Proof: By the result of Proposition 8.3.1, we may find a conformal metric ρ on U such that $\exists\beta > 0$ satisfying $K_\rho(U) \subseteq \mathbb{R}_{\le -\beta}$. Then by the aforementioned generalization of Liouville (Corollary 8.3.3.2), f exhibits constancy on $\mathbb{C}$ and the assertion follows. $\square$

Remark: This is commonly stated in its contrapositive: the image of any non-constant entire function omits at most one value.

8.4 A Spherical Generalization of Normal Families

Picard's Great Theorem requires a more profound concept by generalizing normal families in the one-point compactification of $\mathbb{C}$.

Definition 8.4.1: Let $\{f_n(z)\}$ be a (not necessarily analytic) complex function sequence on a connected set $\Omega \subseteq \mathbb{C}$. If $\forall K \subset \Omega$ compact, $\forall R > 0, \exists N \in \mathbb{N}$ such that $\forall n > N$, $\forall z \in K$, $|f_n(z)| > R$, then $f_n \rightrightarrows \infty$ *locally uniformly spherically on* Ω.

When the "locally uniform limit" is taken to be ∞, the condition of ε-closeness is instead replaced by the requirement that the values eventually leave every fixed compact subset of $\mathbb{C}$ (the given definition is equivalent to: $\forall K \subset \Omega$ compact, $\forall L \subset \mathbb{C}$ compact, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $\forall z \in K$, $f_n(z) \notin L$). In this way, convergence to infinity is treated symmetrically with convergence to finite values by working in the Riemann sphere $\hat{\mathbb{C}}$, where ∞ is simply another accumulation point.

By equipping the extended complex plane $\hat{\mathbb{C}}$ with the spherical metric instead of the Euclidean metric, convergence to ∞ can be treated like convergence to any finite point. In this setting, ∞ is simply another accumulation point, so there is no need to handle it differently from other values.

Let $\{a_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathbb{C}}$ be a sequence. Then we say $a_n \to a_\infty$ *spherically* iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $d_\sigma(a_n, a_\infty) < \varepsilon$, where d_σ is the spherical distance.

Definition 8.4.2: A family of meromorphic functions $\mathcal{F}$ on some $\Omega \subseteq \mathbb{C}$ is said to be *spherically normal* iff every sequence has a locally uniformly spherically convergent subsequence on Ω.

Montel's Theorem for holomorphically normal families in Theorem 5.2.3 can be generalized via the spherical metric by the statement of Marty's Criterion (Theorem 8.4.1).

Definition 8.4.3 *(Spherical Derivative)*: Let $\Omega \subseteq \mathbb{C}$ be an open region or domain. Suppose $f : \Omega \to \hat{\mathbb{C}}$ is meromorphic. Then the *spherical derivative* of f is given by

$$f^\sharp(z) = f^*\sigma(z) = \frac{2|f'(z)|}{1 + |f(z)|^2}$$

for $f(z) \neq \infty$ and

$$f^\sharp(z) = \lim_{\zeta \to z} f^\sharp(\zeta)$$

otherwise.

Proposition 8.4.1: Any linear fractional transformation is spherically uniformly continuous on $\mathbb{C}$.

Proof: Let $\psi(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$. Then,

$$\psi'(z) = \frac{ad - bc}{(cz + d)^2}.$$

The spherical distance between two points $w_1 = \psi(z_1), w_2 = \psi(z_2)$ is given by

$$d_\sigma(w_1, w_2) = \inf_\gamma \int_\gamma \psi^\sharp(z)|\mathrm{d}z| = \inf_\gamma \int_\gamma \frac{2\left|\frac{ad-bc}{(cz+d)^2}\right|}{1 + \left|\frac{az+b}{cz+d}\right|^2}|\mathrm{d}z|$$

where γ joins z_1 and z_2. The spherical distance is bounded by the integral over the Euclidean straight line γ' joining z_1 and z_2:

$$d_\sigma(w_1, w_2) \leq \int_{\gamma'} \frac{2|ad-bc|}{|cz+d|^2+|az+b|^2}|\mathrm{d}z|.$$

Since $\frac{2|ad-bc|}{|cz+d|^2+|az+b|^2} \to 0$ as $z \to \infty$ and $z \mapsto \frac{2|ad-bc|}{|cz+d|^2+|az+b|^2} \in C^0(\mathbb{C})$, it is bounded by some constant M on $\mathbb{C}$. Hence, we have

$$d_\sigma(w_1, w_2) \leq M|z_1 - z_2|.$$

Hence, $\forall \varepsilon > 0$, $\forall |z_1 - z_2| < \frac{\varepsilon}{M}$,

$$d_\sigma(\psi(z_1), \psi(z_2)) < \varepsilon. \qquad \square$$

Proposition 8.4.2: Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of holomorphic functions on a domain $\Omega \subseteq \mathbb{C}$. If $f_n \rightrightarrows f$ locally uniformly spherically, then f is either holomorphic on Ω or identically ∞.

Proof: A result analogous to Theorem 2.3.5 can be used to show that f is spherically continuous. Let $z \in \Omega$ be arbitrary.

1 If $f(z) \neq \infty$, then by spherical continuity, $\exists \delta > 0$ such that $\forall \zeta \in D(z, \delta)$,

$$d_\sigma(f(\zeta), f(z)) < \frac{1}{2} d_\sigma(\infty, f(z)).$$

Similarly, $\exists N \in \mathbb{N}$ such that $\forall n > N$,

$$d_\sigma(f(\zeta), f_n(\zeta)) < \frac{1}{2} d_\sigma(\infty, f(z)).$$

Hence, we have

$$d_\sigma(\infty, f(z)) - d_\sigma(f(z), f_n(\zeta)) > 0.$$

By the reverse triangle inequality, we have

$$d_\sigma(\infty, f_n(\zeta)) > 0.$$

By Weierstrass (Theorem 4.1.1), f is holomorphic on $D(z, \delta)$.

2 Consider $f(z) = \infty$. Assume, for the sake of contradiction, z is an isolated pole of f. Hence, $\exists \delta$ such that f is holomorphic on $D^*(z, \delta)$.

Because each f_n is holomorphic on $D(z, \delta)$, by the Maximum Modulus Principle (Theorem 3.4.1), $\forall n \in \mathbb{N}$,

$$|f_n(\zeta)| \leq \sup_{\xi \in \partial D(z,\delta)} |f_n(\xi)| \quad \forall \zeta \in D(z, \delta).$$

By letting $n \to \infty$, we have

$$|f(\zeta)| \leq \sup_{\xi \in \partial D(z,\delta)} |f(\xi)| < \infty \quad \forall \zeta \in D(z, \delta),$$

contradicting the assumption that $f(z) = \infty$ is an isolated pole. Hence, z must be an accumulation of values evaluating to ∞. By spherical continuity, $\exists \delta > 0$ such that

$$d_\sigma(f(\zeta), \infty) < \frac{\pi}{2} \quad \forall \zeta \in D(z, \delta).$$

Similarly, $\exists N \in \mathbb{N}$ such that $\forall n > N$,

$$d_\sigma(f(\zeta), f_n(\zeta)) < \frac{\pi}{2}.$$

Hence, we have

$$\pi - d_\sigma(\infty, f_n(\zeta)) = d_\sigma(\infty, 0) - d_\sigma(\infty, f_n(\zeta)) > 0.$$

By the reverse triangle inequality, we have

$$d_\sigma(0, f_n(\zeta)) > 0.$$

Hence each $\frac{1}{f_n}$ is holomorphic on $D(z, \delta)$ and converges locally uniformly spherically to $\frac{1}{f}$ on $D(z, \delta)$. By Weierstrass (Theorem 4.1.1), $\frac{1}{f}$ is holomorphic on $D(z, \delta)$ and has zeros that accumulate at z. By the Identity Theorem, $\frac{1}{f} \equiv 0 \Longrightarrow f \equiv \infty$ on $D(z, \delta)$.

Let S be the set of all $z \in \Omega$ such that $f(z)$ is finite. By the argument above, S is open. The complement $\Omega \setminus S$ then consists of all points where $f(z) = \infty$. By the argument above, $\Omega \setminus S$ is also open. Since Ω is connected, by Theorem 3.2.1.3, either $S = \varnothing$ or $S = \Omega$. In the former case, $f \equiv \infty$ on Ω, and in the latter case, f is holomorphic on Ω. □

Theorem 8.4.1 *(Marty's Criterion)*: A family of meromorphic functions $\mathcal{F}$ on some $\Omega \subseteq \mathbb{C}$ is spherically normal iff

$$\{f^\sharp : f \in \mathcal{F}\},$$

or the family of spherical derivatives, is locally uniformly bounded in Ω.

Proof: The condition is equivalent to that of

$$\frac{2|f'(z)|}{1 + |f(z)|^2} \leq M \quad \forall f \in \mathcal{F}$$

for all compact $K \subset \Omega$, $\forall z \in K$, where M depends only on K. Under the assumption that this holds, then

$$d_\sigma(f(z_1), f(z_2)) = \inf_\gamma \int_\gamma \mathrm{d}s_\sigma \leq M|z_2 - z_1| \quad \forall f \in \mathcal{F}$$

where γ joins $f(z_1)$ and $f(z_2)$ where $z_1, z_2 \in K$. Hence, $\forall \varepsilon > 0$, $\forall z_1, z_2 \in K$ such that $|z_1 - z_2| < \frac{\varepsilon}{M}$, $d_\sigma(f(z_1), f(z_2)) < \varepsilon$, and hence $\mathcal{F}$ is *uniformly spherically equicontinuous.* Since $d_\sigma \le \pi$ for any two points by geometry of S^2, $\mathcal{F}$ is also *uniformly spherically bounded* (the compactness of S^2). Then the Arzelà–Ascoli Theorem (Theorem 5.2.2) under the spherical metric gives that $\mathcal{F}$ is a normal family.

Conversely, assume for the sake of contradiction that $\mathcal{F}$ is a normal family such that conclusion is not satisfied. Then, $\exists K \subset \Omega$ compact and a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ such that the sequence

$$\left\{\sup_{z\in K} f_n^\sharp(z)\right\}_{n\in\mathbb{N}}$$

tends to ∞ (specifically, suppose that $\forall n \in \mathbb{N}$, $\sup_{z\in K} f_n^\sharp(z) > n$). By normality, we may extract a locally uniformly spherically convergent subsequence $\left\{f_{n_k}\right\}_{k\in\mathbb{N}} \subseteq \{f_n\}_{n\in\mathbb{N}}$. By Theorem 2.3.5 under the spherical metric, the uniform spherical limit of $\left\{f_{n_k}\right\}_{k\in\mathbb{N}}$, f, is spherically continuous on Ω. For every point $z \in \Omega$, there are two possibilities:

1 If $f(z) \ne \infty$, then by continuity, $\exists \delta > 0$ such that $\forall \zeta \in D(z, \delta)$,

$$d_\sigma(f(\zeta), f(z)) < \frac{1}{2} d_\sigma(\infty, f(z)).$$

Similarly, $\exists N \in \mathbb{N}$ such that $\forall k > N$,

$$d_\sigma\left(f(\zeta), f_{n_k}(\zeta)\right) < \frac{1}{2} d_\sigma(\infty, f(z)).$$

Hence, we have

$$d_\sigma(\infty, f(z)) - d_\sigma\left(f(z), f_{n_k}(\zeta)\right) > 0.$$

By the reverse triangle inequality, we have

$$d_\sigma\left(\infty, f_{n_k}(\zeta)\right) > 0.$$

Hence, the meromorphy of each f_{n_k} is actually holomorphy. By continuity, f is locally uniformly bounded on $D(z, \delta)$. Hence, $\left\{f_{n_k}\right\}_{k>N}$ locally uniformly converges on $D(z, \delta)$. By a result of Weierstrass (Theorem 4.1.1), f is holomorphic on $D(z, \delta)$ and the sequence $\left\{f'_{n_k}\right\}_{k>N}$ locally uniformly converges to f' on $D(z, \delta)$.

By holomorphy of f' on $\overline{D\left(z, \frac{\delta}{2}\right)}$, $\exists M' > 0$ such that $\sup_{\zeta \in \overline{D\left(z, \frac{\delta}{2}\right)}} |f'(\zeta)| < M'$. Uniform convergence of $\left\{f'_{n_k}\right\}_{k>N}$ gives the existence of some $N' > N$ such that $\forall k > N'$,

$$\left|f'_{n_k}(\zeta) - f'(\zeta)\right| < 1 \Longrightarrow \left|f'_{n_k}(\zeta)\right| \leq M' + 1 \quad \forall \zeta \in \overline{D\left(z, \frac{\delta}{2}\right)}.$$

Therefore, $\left\{f'_{n_k}\right\}_{k>N}$ is uniformly bounded by

$$M = \max\left(\{M' + 1\} \cup \left\{\sup_{\zeta \in \overline{D\left(z, \frac{\delta}{2}\right)}} \left|f'_{n_k}(\zeta)\right|\right\}_{N<k\leq N'}\right)$$

on this compact disk. Hence, $\forall k > N$,

$$f^\sharp_{n_k}(\zeta) = \frac{2\left|f'_{n_k}(\zeta)\right|}{1 + \left|f_{n_k}(\zeta)\right|^2} \leq 2\left|f'_{n_k}(\zeta)\right| \leq 2M \quad \forall \zeta \in D\left(z, \frac{\delta}{2}\right) \subset \overline{D\left(z, \frac{\delta}{2}\right)}.$$

2 $f(z) = \infty$, then by continuity, $\exists \delta > 0$ such that $\forall \zeta \in D(z, \delta)$,

$$d_\sigma(f(\zeta), \infty) < \frac{\pi}{2}.$$

Similarly, $\exists N \in \mathbb{N}$ such that $\forall k > N$,

$$d_\sigma\left(f(\zeta), f_{n_k}(\zeta)\right) < \frac{\pi}{2}.$$

Hence, we have

$$\pi - d_\sigma\left(\infty, f_{n_k}(\zeta)\right) = d_\sigma(\infty, 0) - d_\sigma\left(\infty, f_{n_k}(\zeta)\right) > 0.$$

By the reverse triangle inequality, we have

$$d_\sigma\left(0, f_{n_k}(\zeta)\right) > 0.$$

Hence, each $g_{n_k} = \frac{1}{f_{n_k}}$ is holomorphic on $D(z, \delta)$. By continuity, $g = \frac{1}{f}$ is locally uniformly bounded on $D(z, \delta)$. It can also be realized that $\left\{g_{n_k}\right\}_{k>N}$ locally uniformly converges on $D(z, \delta)$. By a result of Weierstrass (Theorem 4.1.1), g is holomorphic on $D(z, \delta)$ and the sequence $\left\{g'_{n_k}\right\}_{k>N}$ locally uniformly converges to g' on $D(z, \delta)$.

By holomorphy of g' on $\overline{D\left(z, \frac{\delta}{2}\right)}$, $\exists M' > 0$ such that $\sup_{\zeta \in \overline{D\left(z, \frac{\delta}{2}\right)}} |g'(\zeta)| < M'$. Uniform convergence of $\left\{g'_{n_k}\right\}_{k>N}$ gives the existence of some $N' > N$ such that $\forall k > N'$,

$$\left|g'_{n_k}(\zeta) - g'(\zeta)\right| < 1 \Longrightarrow \left|g'_{n_k}(\zeta)\right| \le M' + 1 \quad \forall \zeta \in \overline{D\left(z, \frac{\delta}{2}\right)}.$$

Therefore, $\left\{g'_{n_k}\right\}_{k>N}$ is uniformly bounded by

$$M = \max\left(\{M'+1\} \cup \left\{\sup_{\zeta \in \overline{D\left(z,\frac{\delta}{2}\right)}} \left|g'_{n_k}(\zeta)\right|\right\}_{N<k\le N'}\right)$$

on this compact disk. Hence, $\forall k > N$,

$$f^\sharp_{n_k}(\zeta) = \frac{2\left|-\frac{g'_{n_k}(\zeta)}{g_{n_k}(\zeta)^2}\right|}{1 + \left|g_{n_k}(\zeta)\right|^{-2}} = \frac{2\left|g'_{n_k}(\zeta)\right|}{\left|g_{n_k}(\zeta)\right|^2 + 1} \le 2\left|g'_{n_k}(\zeta)\right| \le 2M, \quad \forall \zeta \in D\left(z, \frac{\delta}{2}\right).$$

In essence, for any point z, there exists an open disk D_z centered at z on which the spherical derivatives $f^\sharp_{n_k}$ are bounded by some constant M_z for $k > N_z$. By Heine–Borel (Theorem 1.1.3), there exists a finite collection of disks $\left\{D_{z_j}\right\}_{1\le j\le n}$ that cover K. Thus, $\left\{f^\sharp_{n_k}(z)\right\}_{k>N}$ is uniformly bounded on K by $\max_{1\le j\le n} M_{z_j}$, where $N = \max_{1\le j\le n} N_{z_j}$, contradicting the assumption that $\sup_{z\in K} f^\sharp_n(z) > n$ for all $n \in \mathbb{N}$. □

Theorem 8.4.2 *(Fundamental Normality Test)*: Let $\Omega \subseteq \mathbb{C}$ be a region and suppose that $\mathcal{F}$ is a family of holomorphic functions on Ω. If there exist two different points $\alpha, \beta \in \mathbb{C}$ such that $\{\alpha, \beta\} \cap \bigcup_{f\in\mathcal{F}} f(\Omega) = \varnothing$, then $\mathcal{F}$ must be a spherically normal family.

Proof: Map α and β to $0, 1$ by a linear function $\varphi(z) = \frac{z-\alpha}{\beta-\alpha}$. Then the family of holomorphic functions

$$\tilde{\mathcal{F}} = \{\varphi \circ f : f \in \mathcal{F}\}$$

omits 0 and 1 for all $z \in \Omega$.

By Proposition 8.3.1, $\exists \beta > 0$ such that for

$$\rho(z) = \frac{\sqrt{1+|z|^{\frac{1}{3}}}\sqrt{1+|z-1|^{\frac{1}{3}}}}{|z|^{\frac{5}{6}}|z-1|^{\frac{5}{6}}}, \quad \mathrm{d}s^2_\rho = \rho(z)^2|\mathrm{d}z|^2$$

as in (8.4.5),

$$K_\rho(z) \le -\beta \quad \forall z \in \mathbb{C} \setminus \{0, 1\}.$$

Therefore, if we let $\mu = \rho\sqrt{\beta}$, then

$$K_\mu = -\frac{\nabla^2(\log\circ\mu)}{\mu^2} = -\frac{\nabla^2(\log\circ\rho)}{\rho^2\beta} = \frac{K_\rho}{\beta} \le -1 \quad \text{on} \quad \mathbb{C}\setminus\{0,1\} \tag{8.4.1}$$

Let $\zeta \in \Omega$ be arbitrary and let $r = r_\zeta > 0$ satisfy $D(\zeta, r_\zeta) \subseteq \Omega$. By Corollary 8.3.3.1, the pullback of μ from $\mathbb{C}\setminus\{0,1\}$ to $D(\zeta, r_\zeta) \subseteq \Omega$ satisfies

$$f^*\mu(z) \le \lambda_{r_\zeta}(z-\zeta) \implies \mu(f(z))|f'(z)| \le \frac{2r_\zeta}{r_\zeta^2 - |z-\zeta|^2}$$

$\forall z \in D(\zeta, r_\zeta), f \in \tilde{\mathcal{F}}$. Since $\forall w \in \mathbb{C}\setminus\{0,1\}$,

$$\frac{\sigma}{\mu} = \frac{\frac{2}{1+|w|^2}}{\frac{\sqrt{1+|w|^{\frac{1}{3}}}\sqrt{1+|w-1|^{\frac{1}{3}}}}{|w|^{\frac{5}{6}}|w-1|^{\frac{5}{6}}}} \to \begin{cases} 0 & \text{as } w \to 0 \\ 0 & \text{as } w \to 1 \\ \frac{2|w|^{-2}}{|w|^{-\frac{4}{3}}} \to 0 & \text{as } w \to \infty \end{cases}.$$

Hence, there exist open neighborhoods U_0, U_1, U_∞ of $0, 1, \infty$ respectively on which $\frac{\sigma}{\mu} < 1$. Since $\frac{\sigma}{\mu} \in C^0(\mathbb{C})$, by Theorem 1.2.13, $\exists M' > 0$ such that $\frac{\sigma}{\mu} < M'$ on $\mathbb{C}\setminus(U_0 \cup U_1 \cup U_\infty)$. Let $M = \max(M', 1)$, and

$$\therefore \sigma \le M\mu \quad \text{on} \quad \mathbb{C}\setminus\{0,1\}.$$

Hence, $\forall f \in \tilde{\mathcal{F}}$, we have by virtue of (8.4.1),

$$f^\sharp(z) = \sigma\circ f(z)|f'(z)| \le M\mu\circ f(z)|f'(z)| \le \frac{2r_\zeta M}{r_\zeta^2 - |z-\zeta|^2}$$

for any $z \in D(\zeta, r_\zeta)$. Now restricting z to $D\left(\zeta, \frac{r_\zeta}{2}\right)$, we have

$$|z-\zeta|^2 < \frac{r_\zeta^2}{4} \implies r_\zeta^2 - |z-\zeta|^2 > \frac{3r_\zeta^2}{4} \implies \left|f^\sharp(z)\right| < \frac{8r_\zeta M}{3r_\zeta^2} = \frac{8M}{3r_\zeta}.$$

For any compact $K \subset \Omega$, the collection of open disks

$$\left\{D\left(\zeta, \frac{r_\zeta}{2}\right)\right\}_{\zeta\in K}$$

forms an open cover of K. Hence, by Heine–Borel (Theorem 1.1.3), it admits a finite subcover

$$\left\{D\left(\zeta_k, \frac{r_{\zeta_k}}{2}\right)\right\}_{1\le k\le n}$$

for some $n \in \mathbb{N}$. Then $\left\{f^\sharp : f \in \tilde{\mathcal{F}}\right\}$ is uniformly bounded on K by

$$M_K = \max\left\{\frac{8M}{3r_{\zeta_k}} : 1 \le k \le n\right\}$$

and is thus locally uniformly bounded on Ω. Marty's Criterion (Theorem 8.4.1) gives the normality of $\tilde{\mathcal{F}}$; since φ is linear, it follows that $\mathcal{F}$ is also normal on Ω. ☐

Corollary 8.4.2.1 *(Montel–Carathéodory)*: Let $\Omega \subseteq \mathbb{C}$ be a region and suppose that $\mathcal{F}$ is a family of meromorphic functions on Ω. If there exist three different points $\alpha, \beta, \gamma \in \hat{\mathbb{C}}$ such that $\{\alpha, \beta, \gamma\} \cap \bigcup_{f\in\mathcal{F}} f(\Omega) = \varnothing$, then $\mathcal{F}$ must be a spherically normal family.

Proof: Let $\varphi(z) = \frac{(z-\alpha)(\beta-\gamma)}{(z-\gamma)(\beta-\alpha)}$ be a Möbius transformation mapping α, β, γ to $0, 1, \infty$, respectively. Hence, the family of meromorphic functions

$$\tilde{\mathcal{F}} = \{\varphi \circ f : f \in \mathcal{F}\}$$

omits 0, 1, and ∞ (and hence each function is holomorphic). By the Fundamental Holomorphic Normality Test (Theorem 8.4.2), $\tilde{\mathcal{F}}$ is normal.

By Proposition 8.4.1, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall |w_1 - w_2| < \delta$ in $\mathbb{C}$,

$$d_\sigma(\varphi^{-1}(w_1), \varphi^{-1}(w_2)) < \varepsilon.$$

Let $\left\{\tilde{f}_n\right\}_{n\in\mathbb{N}}$ be any function sequence in $\mathcal{F}$ and let $\left\{\tilde{f}_{n_k}\right\}_{k\in\mathbb{N}}$ be locally uniformly convergent to $\tilde{f}$ on a compact set $K \subset \Omega$. Then $\exists N \in \mathbb{N}$ such that $\forall k > N$,

$$\left|\tilde{f}_{n_k}(z) - \tilde{f}(z)\right| < \delta \qquad \forall z \in K.$$

Therefore, $\forall z \in K$, $k > N$, we have

$$d_\sigma\left(\varphi^{-1} \circ \tilde{f}_{n_k}(z), \varphi^{-1} \circ \tilde{f}(z)\right) = d_\sigma\left(f_{n_k}(z), f(z)\right) < \varepsilon.$$

Hence, every sequence f_n has a locally uniformly spherically convergent subsequence, and the normality of $\mathcal{F}$ follows. ☐

8.5 The Great Picard, Bloch, Landau, and Schottky Theorems

Recall the Casorati–Weierstrass Theorem, one of the earliest results on the value distribution near essential singularities:

Theorem 4.2.3

We will now prove a more advanced characterization of this distribution by methods of differential geometry.

Theorem 8.5.1 *(Picard's Great Theorem)*: Suppose f is holomorphic on a punctured neighborhood $D^*(z_0, \delta)$ of z_0. If z_0 is an essential singularity of f, then $f(D^*(z_0, \delta))$ omits at most one value of $\mathbb{C}$.

Proof: Without loss of generality, assume $z_0 = 0$ and that f omits the values 0 and 1 (otherwise, consider $z \mapsto \frac{1}{\beta - \alpha}(f(z + z_0) - \alpha)$, where α and β are the omitted values). Define the family

$$\mathcal{F} = \left\{z \mapsto f\left(\frac{z}{n}\right) : n \in \mathbb{N}\right\}$$

of holomorphic functions on $D^*(0, \delta)$. Since f omits 0 and 1, each element of $\mathcal{F}$ does as well. By the Fundamental Normality Test (Theorem 8.4.2), $\mathcal{F}$ is spherically normal. Thus, there exists a subsequence $\left\{f_{n_k}\right\}_{k \in \mathbb{N}} \subseteq \mathcal{F}$ that converges locally uniformly on $D^*(0, \delta)$ in the spherical metric. By Proposition 8.4.2, this subsequence converges locally uniformly either to a holomorphic function on $D^*(0, \delta)$ or to ∞ thereon.

1 Suppose $\left\{f_{n_k}\right\}_{k \in \mathbb{N}}$ converges locally uniformly to a holomorphic function on $D^*(0, \delta)$. Then $\left\{f_{n_k}\right\}_{k \in \mathbb{N}}$ is uniformly bounded on $\partial D\left(0, \frac{\delta}{2}\right)$. Hence, there exists $M > 0$ such that

$$\left|f\left(\frac{z}{n_k}\right)\right| = \left|f_{n_k}(z)\right| < M \quad \forall z \in \partial D\left(0, \frac{\delta}{2}\right), k \in \mathbb{N}.$$

In other words, f is bounded by M on every circle $\partial D\left(0, \frac{\delta}{2n_k}\right)$ for $k \in \mathbb{N}$. By the Maximum Modulus Principle (Theorem 3.4.1), f is then bounded by M on each annulus $\overline{D\left(0, \frac{\delta}{2n_k}\right) \setminus D\left(0, \frac{\delta}{2n_{k+1}}\right)}$ for $k \in \mathbb{N}$. As

$$\bigcup_{k \in \mathbb{N}} \overline{D\left(0, \frac{\delta}{2n_k}\right) \setminus D\left(0, \frac{\delta}{2n_{k+1}}\right)} = \overline{D\left(0, \frac{\delta}{2n_1}\right)} \setminus \{0\},$$

it follows that f is bounded on $D^*\left(0, \frac{\delta}{2}\right)$. By Riemann's Removable Singularity Theorem (Theorem 3.2.6), f therefore extends holomorphically to 0.

2 Suppose $\left\{f_{n_k}\right\}_{k \in \mathbb{N}}$ converges locally uniformly to ∞ on $D^*(0, \delta)$. Then, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $k > N$,

$$\left|\frac{1}{f\left(\frac{z}{n_k}\right)}\right| = \left|\frac{1}{f_{n_k}(z)}\right| < \varepsilon \quad \forall z \in \partial D\left(0, \frac{\delta}{2}\right).$$

By the same reasoning as in the previous case, $\left|\frac{1}{f}\right| < \varepsilon$ on

$$\bigcup_{k>N} \overline{D\left(0, \frac{\delta}{2n_k}\right)} \setminus D\left(0, \frac{\delta}{2n_{k+1}}\right) \supset D^*\left(0, \frac{\delta}{2n_{N+1}}\right).$$

Thus, by the definition of the limit, $\lim_{z \to 0} \frac{1}{f(z)} = 0$, so f has a pole at 0.

In either case, we have derived a meromorphic continuation of f to 0, contradicting the assumption that 0 is an essential singularity of f. ☐

Corollary 8.5.1.1: Suppose that f is meromorphic on a punctured neighborhood $D^*(z_0, \delta)$ of z_0. If $f(D^*(z_0, \delta))$ omits at least three different values of $\hat{\mathbb{C}}$, then f has a meromorphic continuation to z_0.

Proof: A linear fractional transformation maps the omitted values to 0, 1, and ∞, mapping f so that it exhibits holomorphy. Similar to Corollary 8.4.2.1, the preceding result is preserved under the inverse linear fractional transformation. ☐

Remark: An accumulation point of poles is an essential singularity on the Riemann sphere.

Picard's Great Theorem is also a generalization of Picard's Little Theorem (Theorem 8.3.4):

Proof: Let $g(z) = f\left(\frac{1}{z}\right)$ with an isolated singularity at 0 and a removable singularity at ∞. By Picard's Great Theorem (Theorem 8.5.1), $g(z)$ has a meromorphic extension to $z = 0$. If $z = 0$ is removable, by virtue of Proposition 4.3.1 and Theorem 3.2.3, the constancy of g and f follows.

If instead $z = 0$ is a pole of g, then $z = \infty$ is a pole of f, and hence f is a polynomial. Assume, for the sake of contradiction that f is non-constant. Then $\forall w \in \mathbb{C}$, the Fundamental Theorem of Algebra (Theorem 3.3.1) gives the existence of some $z \in \mathbb{C}$ such that $f(z) = w$. Hence, f attains every value $w \in \mathbb{C}$. This contradicts the statement and hence f is constant. ☐

The efforts of many mathematicians resulted in several alternative proofs following that of Picard; the geometric realization of Ahlfors (Theorem 8.3.1) was followed by results discovered by R. M. Robinson. Other approaches from Nevanlinna theory appeared later in the 20th century.

Picard's original proof, providing an advanced characterization of the value distribution at essential singularities, relied primarily on the properties of the elliptic modular function (as a "covering map"). From this, Picard deduced that the function would necessarily extend holomorphically across the singularity, contradicting its essential nature. Thus, his proof established that near an essential singularity, a holomorphic function attains every complex value, with at most one exception, infinitely often.

More importantly, we have shown the utility of even seemingly fundamental differential geometry, which can also be used in the proof of many other important results.

The methods of differential geometry can also be used to prove the statements of the following theorems (which can also be independently used to prove the Picard theorems), but is made meaningful with the notion of Riemann surfaces.

Theorem 8.5.2 *(Bloch)*: Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic such that $|f'(0)| = 1$. Then there is a region $S \subseteq \mathbb{D}$ on which f is univalent such that $f(S)$ contains a disk with a radius of at least $\frac{\sqrt{3}}{4}$ (known as a schlicht disk).

Remark: *Bloch's constant* B is defined as the supremum of the radii of such disks that can be contained in $f(\mathbb{D})$ for any holomorphic function $f : \mathbb{D} \to \mathbb{C}$ satisfying $f'(0) = 1$.

The precise value of B remains unknown to this day. In 1937, H. Grunsky and L. Ahlfors established the bound

$$B \leq \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{\frac{\sqrt{3}-1}{2}},$$

where Γ denotes the Gamma function (as in @ eq:gammafunction). Later the lower bound of $\frac{\sqrt{3}}{4}$ was given, then to be refined to $B \geq \frac{\sqrt{3}}{4} + \frac{10^{-12}}{13}$ by M. Bonk, which was further improved to $B \geq \frac{\sqrt{3}}{4} + \frac{1}{5000}$ in 1996 by H. Chen and P. M. Gauthier.

Grunsky and Ahlfors actually conjectured that the upper bound in their inequality is exact – that is, $B = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{\frac{\sqrt{3}-1}{2}}$.

Theorem 8.5.3 *(Landau)*: The image of any holomorphic function f in $\mathbb{D}$ satisfying $f(0) = 0$ and $f'(0) = 1$ contains a disk with radius of at least $\frac{1}{2}$.

Remark: Similarly, the estimate $\frac{1}{2}$ is not optimal. It was established that the corresponding *Landau's constant* lies between $\frac{1}{2}$ and $\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}\right)}$.

Without Riemann surfaces, the proof of the two aforesaid results are rather difficult, as a distinction must be established for a point $w \in f(\mathbb{D})$ which two values z_1, z_2 map to. Otherwise, when we describe a schlicht disk at a point in the image, we may be talking about different "sheets" or "branches," although each fixed sheet may describe perfectly well-defined analytic functions, although they describe different "copies." More details may be found in [2].

Hence, for simplicity, we entertain a much simpler case without algebraic branch points.

Theorem 8.5.4: Let $f : \mathbb{D} \to \mathbb{C}$ be univalent such that $|f'(0)| = 1$. Then $f(\mathbb{D})$ contains a disk with a radius of at least $\frac{\sqrt{3}}{4}$.

Proof: For $w \in f(\mathbb{D})$, let $\phi(w)$ denote the radius of the largest schlicht disk in $f(\mathbb{D})$ centered at w (it is mapped to univalently by f on some subdomain). Trivially, ϕ is C^0 and vanishes toward the boundary of $f(\mathbb{D})$.

Define the metric

$$\rho(w) = \frac{A}{\sqrt{\phi(w)}(A^2 - \phi(w))}, \quad \mathrm{d}s = \rho(w)|\mathrm{d}w|$$

for $w \in f(\mathbb{D})$ and $\rho(w) \neq 0$, where $A^2 > \sup_w \phi(w)$ is a constant. We may assume that $\sup_w \phi(w)$ is finite, since otherwise the theorem is already proved for f.

For every point $w_0 \in f(\mathbb{D})$, the bounding circle corresponding to $\phi(w_0)$ passes through a (at least one) boundary point, denoted by $b = b_{w_0}$. Let $\phi_{w_0}(w) = |w - b|$ and let

$$\rho_{w_0}(w) = \frac{A}{\sqrt{\phi_{w_0}(w)}\left(A^2 - \phi_{w_0}(w)\right)}, \quad w \in D(w_0, \phi(w_0)).$$

By the definition of ϕ, we have $\phi_{w_0} \geq \phi$ everywhere in this neighborhood. Since $\rho_{w_0} = \left(\phi_{w_0} \mapsto \sqrt{\phi_{w_0}}\right)^* \left(\varphi \mapsto \frac{2A}{A^2 - \varphi^2}\right)$ is the pullback of the hyperbolic metric in (8.5.4), the metric $\rho_{w_0}(w)$ has the constant negative curvature of -1.

Our goal is to construct ρ_{w_0} so that it is the function of a supporting metric for ρ (satisfies the criteria for Theorem 8.3.3). For $\rho_{w_0} \leq \rho$ to be satisfied, we consider

$$\sqrt{\phi_{w_0}(w)}\left(A^2 - \phi_{w_0}(w)\right) \geq \sqrt{\phi(w)}(A^2 - \phi(w)), \quad \phi_{w_0}(w) \geq \phi(w).$$

In particular, we want

$$t \mapsto \sqrt{t}(A^2 - t)$$

to be increasing on $[0, \phi(w_0) + \delta]$ for arbitrary $\delta > 0$. The function itself can be calculated to be increasing for $t \leq \frac{A^2}{3}$ by elementary methods (using derivative tests). Therefore, the conditions for a supporting metric are satisfied if $\frac{A^2}{3} \geq \sup_{w \in f(\mathbb{D})} \phi(w) + \delta \geq \phi(w_0) + \delta$. Without loss of generality we let $\delta \to 0^+$ and thus, under the condition that $\frac{A^2}{3} > \sup_w \phi(w)$, Theorem 8.3.3 gives that

$$\rho(w)|\mathrm{d}w| \leq \frac{2|\mathrm{d}z|}{1 - |z|^2}.$$

Let $z = 0$, $w = f(0)$, so that by the theorem conditions, $\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = 1$, and therefore

$$\rho(f(0)) \leq 2 \Longrightarrow A \leq 2\sqrt{\phi(f(0))}\left[A^2 - \phi(f(0))\right].$$

By the previous assumptions on A, the corresponding function on the right-hand side is increasing, and since $\phi(f(0)) \leq \sup_w \phi(w)$, we have

$$A^2 \leq 4 \sup_w \phi(w) \left(A^2 - \sup_w \phi(w)\right)^2.$$

As $\frac{A^2}{3} \to \sup_w \phi(w)^+$ (A was chosen arbitarily, so this is valid), it follows that

$$\sup_w \phi(w) \geq \frac{\sqrt{3}}{4}. \qquad \square$$

It is however notable that the proof follows similarly for general functions, but instead we consider functions $f : \mathbb{D} \to W$, where W is a Riemann surface and the "singularities" are not only boundary points but also algebraic branch points (where $f' = 0$).

Theorem 8.5.5 *(Landau–Carathéodory)*: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $a_1 \neq 0$ and f is holomorphic on $D(0, r)$. If f omits 0 and 1, then $\exists R$ dependent only on a_0 and a_1 such that $r \leq R$.

Theorem 8.5.6 *(Schottky)*: Suppose that $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and omits 0 and 1. Then

$$\log|f(z)| < \frac{1 + |z|}{1 - |z|}\left(7 + \log^+|f(0)|\right),$$

where $\log^+(x) = \max\{0, \log x\}$ (common notation in Nevanlinna theory).

Proof: Consider conformal map $\zeta_1(w) : \mathbb{C} \setminus [0, 1] \to \mathbb{C} \setminus \overline{\mathbb{D}}$, which extends to 0, 1, and ∞ continuously such that $\zeta_1(\infty) = \infty$, $\zeta_1(1) = 1$, $\zeta_1(0) = -1$. Explicitly, we have the relationship

$$\zeta_1 + \frac{1}{\zeta_1} = 4w - 2$$

as an affine transformation of the inverse *Joukowski transform* (inverse of $z \mapsto z + z^{-1}$). The solution is given by

$$\zeta_1 = (2w - 1) + 2w\sqrt{1 - \frac{1}{w}} = (2w - 1) + 2w \exp\left[\frac{1}{2} \operatorname{Log}\left(1 - \frac{1}{w}\right)\right],$$

where the branch cut of the square root is taken to be the negative real axis, which maps to $[0, 1]$ in terms of w (and with the principal branch logarithm). Moreover, this explicit map maps $[0, 1]$ to $\partial\mathbb{D}$, since for $w \in [0, 1]$, the term involving the square root is purely imaginary thus $|\zeta_1| = \sqrt{4w^2 + 1 - 4w - 4w^2 + 4w} = 1$. Because as $w \to \infty$, $\zeta_1 \to \infty$, ζ_1 here is a valid conformal map. □

9 A Glimpse into the Function Theory of Multiple Complex Variables

As in the single-variable case, there exist natural extensions of the definitions of holomorphy and of derivatives, as well as direct analogs of integral theorems.

Many other results, however, have to be separately derived.

Throughout mathematical history, many efforts were made to study the nature of complex functions in a multivariate setting. In the 20th century, Poincaré proved that the unit ball $B^n = \left\{(z_1, ..., z_n) \in \mathbb{C}^n : \sum_{k=1}^{n} |z_k|^2 < 1\right\}$ and the polydisk $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$ are not biholomorphically equivalent by comparison of their automorphism groups, under certain assumptions of the automorphisms on their respective boundaries; the proof was later formalized by Cartan, but is nonetheless largely attributed to Poincaré. As one of the first of many deviations, the miracle of the Riemann Mapping Theorem (Theorem 5.3.1) fails in higher dimensions. Hartogs later also showed that poles and essential singularities cannot exist as isolated singularities of multivariate holomorphic functions. Perhaps these are the results of an unsatisfactory generalization.

Whereas the efforts of mathematicians of over two centuries give rise to the development of function theory of one complex variable, the theory of multivariate complex functions is still largely rudimentary. Many seemingly fundamental problems still largely remain as conjecture.

9.1 Consequences of Holomorphy

Obviously, we will first formally define the concept of holomorphy in higher dimensions.

Definition 9.1.1: A function $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}$ is *holomorphic* if it is holomorphic in each variable when the others are held constant.

If we consider f to be a function of $z_1, \overline{z_1}, z_2, \overline{z_2}, ..., z_n, \overline{z_n}$, then f is holomorphic iff $\frac{\partial f}{\partial \overline{z_k}} \equiv 0$ for all $1 \leq k \leq n$ and f has all continuous partial derivatives.

Theorem 9.1.1 (*Cauchy's Integral Formula on Polydisks*): Fix $\boldsymbol{a} = (a_1, ..., a_n) \in \mathbb{C}^n$ arbitrarily and suppose $r_1, r_2, ..., r_n > 0$ are the radii of the polydisk defined by $\Omega = \prod_{k=1}^n D(a_k, r_k)$ (where the product here is the Cartesian product). Suppose $f : \overline{\Omega} \to \mathbb{C}$ is holomorphic. For fixed $k_1, k_2, ..., k_n \in \mathbb{Z}_{\geq 0}$, we have that

$$\left(\prod_{j=1}^{n} \frac{\partial^{k_j}}{\partial z_j^{k_j}}\right) f(\boldsymbol{z}) = \frac{\prod_{j=1}^n k_j!}{(2\pi \mathrm{i})^n} \oint_{\partial D(a_1,r_1)} \cdots \oint_{\partial D(a_n,r_n)} \frac{f(\zeta_1, ..., \zeta_n) \mathrm{d}\zeta_n \cdots \mathrm{d}\zeta_1}{\prod_{j=1}^n \left(\zeta_j - z_j\right)^{k_j+1}}$$

for any $z = (z_1, z_2, ..., z_n) \in \Omega$.

Proof: By Cauchy–Goursat (Theorem 3.2.1), we have

$$\frac{\partial^{k_1}}{\partial z_1^{k_1}} f(\boldsymbol{z}) = \frac{k_1!}{2\pi \mathrm{i}} \oint_{\partial D(a_1,r_1)} \frac{f(\zeta_1, z_2, ..., z_n)}{(\zeta_1 - z_1)^{k_1+1}} \mathrm{d}\zeta_1$$

for $z \in \Omega$, which is holomorphic. Thus, by the same application on $\frac{\partial^{k_1}}{\partial z_1^{k_1}} f(\boldsymbol{z})$, we have

$$\frac{\partial^{k_2}}{\partial z_2^{k_2}} \frac{\partial^{k_1}}{\partial z_1^{k_1}} f(\boldsymbol{z}) = \frac{k_2! k_1!}{(2\pi \mathrm{i})^2} \oint_{\partial D(a_2,r_2)} \oint_{\partial D(a_1,r_1)} \frac{f(\zeta_1, \zeta_2, z_3, ..., z_n) \mathrm{d}\zeta_1 \mathrm{d}\zeta_2}{(\zeta_1 - z_1)^{k_1+1} (\zeta_2 - z_2)^{k_2+1}}.$$

By reiterating n times and reversing the order of differentiation and integration, the conclusion follows. □

By the boundedness assumption for f, we have:

Corollary 9.1.1.1 (*Cauchy's Estimate on Polydisks*): Let $\boldsymbol{a} = (a_1, a_2, ..., a_n) \in \mathbb{C}^n$ be fixed and suppose $r_1, r_2, ..., r_n > 0$ are the radii of the polydisk defined by $\Omega = \prod_{k=1}^n D(a_k, r_k)$ (where the product here is the Cartesian product). Suppose $f : \overline{\Omega} \to \mathbb{C}$ is holomorphic. For fixed $k_1, k_2, ..., k_n \in \mathbb{Z}_{\geq 0}$, we have that

$$\left| \left(\prod_{j=1}^{n} \frac{\partial^{k_j}}{\partial z_j^{k_j}}\right) f(\boldsymbol{z}) \right| \leq \prod_{j=1}^{n} \left(\frac{k_j!}{r_j^{k_j}}\right) \sup_{\boldsymbol{\zeta} \in \prod_{j=1}^n \partial D(a_j, r_j)} |f(\boldsymbol{\zeta})|$$

for any $\boldsymbol{z} = (z_1, z_2, ..., z_n) \in \Omega$.

Proof: For each j, let ε_j satisfy $D(z_j, \varepsilon_j) \subseteq D(a_j, r_j)$. By Cauchy's Integral Formula (Theorem 9.1.1), we have

$$\left|\left(\prod_{j=1}^{n}\frac{\partial^{k_j}}{\partial z_j^{k_j}}\right)f(\boldsymbol{z})\right|$$
$$\leq \frac{\prod_{j=1}^{n}k_j!}{(2\pi)^n}\oint_{\partial D(a_1,r_1)}\cdots\oint_{\partial D(a_n,r_n)}\left|\frac{f(\zeta_1,\ldots,\zeta_n)}{\prod_{j=1}^{n}\left(\zeta_j-z_j\right)^{k_j+1}}\right||\mathrm{d}\zeta_n|\cdots|\mathrm{d}\zeta_1|.$$

For each j, let $\zeta_j = a_j + r_j\mathrm{e}^{\mathrm{i}t_j}$, and it follows that $\mathrm{d}\zeta_j = \mathrm{i}r_j\mathrm{e}^{\mathrm{i}t_j}\,\mathrm{d}t_j$. Because $\left|\zeta_j - z_j\right| > \varepsilon_j$, we have, after substitution,

$$\begin{aligned}\left|\left(\prod_{j=1}^{n}\frac{\partial^{k_j}}{\partial z_j^{k_j}}\right)f(\boldsymbol{z})\right| &\leq \frac{\prod_{j=1}^{n}\left(r_jk_j!\right)}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\left|\frac{f(\zeta_1,\ldots,\zeta_n)}{\prod_{j=1}^{n}\varepsilon_j^{k_j+1}}\right|\mathrm{d}t_n\cdots\mathrm{d}t_1\\ &\leq \prod_{j=1}^{n}\left(\frac{r_jk_j!}{2\pi\varepsilon_j^{k_j+1}}\right)\sup_{\boldsymbol{\zeta}\in\prod_{j=1}^{n}\partial D(a_j,r_j)}|f(\boldsymbol{\zeta})|\int\cdots\int_{[0,2\pi]^n}\mathrm{d}t_n\cdots\mathrm{d}t_1\\ &\leq \prod_{j=1}^{n}\left(\frac{k_j!}{r_j^{k_j}}\right)\sup_{\boldsymbol{\zeta}\in\prod_{j=1}^{n}\partial D(a_j,r_j)}|f(\boldsymbol{\zeta})|,\end{aligned}$$

since $\varepsilon_j \leq r_j$ for all j. □

Similar to the univariate case, there are Taylor expansions of holomorphic functions in several complex variables.

Theorem 9.1.2: Let $f : \mathbb{C}^n \to \mathbb{C}$ be holomorphic on (a neighborhood of) the closure $\overline{\Omega}$ of a polydisk $\Omega = \prod_{k=1}^{n} D(a_k, r_k)$ centered at $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$. Then, for any $\boldsymbol{z} = (z_1, z_2, \ldots, z_n) \in \Omega$, we have the expansion

$$f(\boldsymbol{z}) = \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}a_{k_1,\ldots,k_n}(z_1-a_1)^{k_1}\cdots(z_n-a_n)^{k_n}, \qquad (9.1.1)$$

where $\forall k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$,

$$a_{k_1,\ldots,k_n} = \frac{1}{\prod_{j=1}^{n}k_j!}\left(\prod_{j=1}^{n}\frac{\partial^{k_j}}{\partial z_j^{k_j}}\right)f(\boldsymbol{a}).$$

The series converges absolutely and uniformly on Ω.

Proof: By Theorem 9.1.1 we have

$$f(\boldsymbol{z}) = \frac{1}{(2\pi\mathrm{i})^n}\oint_{\partial D(a_1,r_1)}\cdots\oint_{\partial D(a_n,r_n)}\frac{f(\boldsymbol{\zeta})}{(\zeta_1-z_1)\cdots(\zeta_n-z_n)}\mathrm{d}\zeta_n\cdots\mathrm{d}\zeta_1.$$

For each j, since $\left|z_j - a_j\right| < r_j = \left|\zeta_j - a_j\right|$ on $\partial D(a_j, r_j)$, the geometric series expansion holds:

$$\frac{1}{\zeta_j - z_j} = \frac{1}{\zeta_j - a_j} \cdot \frac{1}{1 - \frac{z_j - a_j}{\zeta_j - a_j}} = \sum_{k_j=0}^{\infty} \frac{\left(z_j - a_j\right)^{k_j}}{\left(\zeta_j - a_j\right)^{k_j+1}},$$

which converges uniformly in ζ_j on $\partial D(a_j, r_j)$. Hence, we have

$$f(\boldsymbol{z}) = \frac{1}{(2\pi \mathrm{i})^n} \sum_{k_1=0}^{\infty} \oint_{\partial D(a_1, r_1)} \cdots \oint_{\partial D(a_n, r_n)} \frac{f(\boldsymbol{\zeta})(z_1 - a_1)^{k_1} \mathrm{d}\zeta_n \cdots \mathrm{d}\zeta_1}{(\zeta_1 - a_1)^{k_1+1}(\zeta_2 - z_2) \cdots (\zeta_n - z_n)}$$

where uniform convergence has allowed the interchange of summation and integration. Reiteration of this process gives

$$f(\boldsymbol{z}) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \frac{\prod_{j=1}^{n} \left(z_j - a_j\right)^{k_j}}{(2\pi \mathrm{i})^n} \oint_{\partial D(a_1, r_1)} \cdots \oint_{\partial D(a_n, r_n)} \frac{f(\boldsymbol{\zeta}) \mathrm{d}\zeta_n \cdots \mathrm{d}\zeta_1}{\prod_{j=1}^{n} \left(\zeta_j - a_j\right)^{k_j+1}}.$$

By the Cauchy Integral Formula (Theorem 9.1.1), we have

$$\left(\prod_{j=1}^{n} \frac{\partial^{k_j}}{\partial z_j^{k_j}}\right) f(\boldsymbol{a}) = \frac{\prod_{j=1}^{n} k_j!}{(2\pi \mathrm{i})^n} \oint_{\partial D(a_1, r_1)} \cdots \oint_{\partial D(a_n, r_n)} \frac{f(\boldsymbol{\zeta}) \mathrm{d}\zeta_n \cdots \mathrm{d}\zeta_1}{\prod_{j=1}^{n} \left(\zeta_j - a_j\right)^{k_j+1}},$$

and hence if we let

$$a_{k_1, \dots, k_n} = \frac{1}{\prod_{j=1}^{n} k_j!} \left(\prod_{j=1}^{n} \frac{\partial^{k_j}}{\partial z_j^{k_j}}\right) f(\boldsymbol{a}),$$

then (9.1.1) follows. Cauchy's Estimate (Corollary 9.1.1.1) gives that

$$\left|a_{k_1, \dots, k_n}\right| \leq M \prod_{j=1}^{n} \left(\frac{1}{\rho_j^{k_j}}\right),$$

where $M = \sup_{\boldsymbol{\zeta} \in \prod_{j=1}^{n} \partial D(a_j, r_j)} |f(\boldsymbol{\zeta})|$ for some $\rho_j > r_j$ for all j. Hence,

$$\begin{aligned}\left|\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}a_{k_1,\dots,k_n}\prod_{j=1}^{n}\left(z_j-a_j\right)^{k_j}\right| &\leq \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\left|a_{k_1,\dots,k_n}\right|\prod_{j=1}^{n}\left|z_j-a_j\right|^{k_j}\\ &\leq M\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\prod_{j=1}^{n}\left|\frac{r_j}{\rho_j}\right|^{k_j}\\ &= M\prod_{j=1}^{n}\sum_{k_j=0}^{\infty}\left|\frac{r_j}{\rho_j}\right|^{k_j} < \infty.\end{aligned}$$

By the Weierstrass M–Test (Theorem 2.3.2), the series converges absolutely and uniformly on Ω. □

Theorem 9.1.3 *(Identity)*: Let f be a holomorphic function on $\Omega \subseteq \mathbb{C}^n$. If the set $\{z \in \Omega : f(z) = 0\}$ has an accumulation point in Ω, then $f \equiv 0$ on Ω.

Theorem 9.1.4 *(Maximum Modulus Principle)*: Let $\Omega \subset \mathbb{C}^n$ be a open bounded region, and suppose that $f : \Omega \to \mathbb{C}$ is holomorphic. If

$$M = \sup_{\zeta\in\partial\Omega}\lim_{\substack{z\to\zeta\\ z\in\Omega}}|f(z)|,$$

then $|f(z)| < M$ for all $z \in \Omega$, unless f is constant.

Theorem 9.1.5 *(Weierstrass)*: Suppose that $\Omega \subseteq \mathbb{C}^n$ is a region and that $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of holomorphic functions $\Omega \to \mathbb{C}$. If $\{f_k\}_{k\in\mathbb{N}}$ converges locally uniformly to f on Ω, then f is holomorphic on Ω. Moreover, $\forall k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$,

$$\left(\prod_{j=1}^{n}\frac{\partial^{k_j}}{\partial z_j^{k_j}}\right)f_k \rightrightarrows \left(\prod_{j=1}^{n}\frac{\partial^{k_j}}{\partial z_j^{k_j}}\right)f$$

on compact subsets of Ω.

Theorem 9.1.6 *(Montel)*: A family $\mathcal{F}$ of holomorphic functions on some region $\Omega \subseteq \mathbb{C}^n$ is normal iff it is locally uniformly bounded on Ω.

9.2 The Group of Holomorphic Automorphisms on $\mathbb{D}^n$ and B^n

A function $\boldsymbol{f} : \Omega \subseteq \mathbb{C}^m \to \mathbb{C}^n$ is called *holomorphic* iff each of its component functions is holomorphic. It is important to allow for vector-valued outputs, since we are interested in automorphisms on complex domains in higher dimensions.

For the aforesaid purpose, we require a generalization of the Schwarz Lemma (Lemma 3.5.1), which is equivalent to several results of Cartan.

In preparation, we will introduce several relevant concepts.

Definition 9.2.1 *(Multi-Index Notation)*: A *multi-index* is an n-tuple of non-negative integers $\boldsymbol{k} = (k_1, ..., k_n) \in \mathbb{Z}^n_{\geq 0}$. We define

$$|\boldsymbol{k}| = \sum_{j=1}^{n} k_j, \quad \boldsymbol{z}^{\boldsymbol{k}} = \prod_{j=1}^{n} z_j^{k_j}, \quad \partial^{\boldsymbol{k}} = \frac{\partial^{|\boldsymbol{k}|}}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} = \prod_{j=1}^{n} \frac{\partial^{k_j}}{\partial z_j^{k_j}},$$

where $\boldsymbol{z} = (z_1, ..., z_n) \in \mathbb{C}^n$.

Definition 9.2.2: A polynomial $\psi : \mathbb{C}^n \to \mathbb{C}^m$ of several variables is said to be *homogeneous of degree d* iff

$$\psi(\lambda \boldsymbol{z}) = \lambda^d \psi(\boldsymbol{z}) \quad \forall \lambda \in \mathbb{C}, \boldsymbol{z} \in \mathbb{C}^n,$$

or equivalently, iff ψ can be written as

$$\psi(\boldsymbol{z}) = \sum_{|\boldsymbol{k}|=d} \boldsymbol{a}_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}$$

where $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$ is a multi-index.

Proposition 9.2.1: Let $\psi : \mathbb{C}^n \to \mathbb{C}^m$ be a homogeneous polynomial of degree d.

1 For any multi-index $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ with $\|\boldsymbol{\alpha}\| = r \leq d$,

$$\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{z}) = \frac{\partial^r \psi}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(\boldsymbol{z})$$

is a homogeneous polynomial of degree $d - r$.

2 If $r = d \neq 0$, then $\partial^{\boldsymbol{\alpha}} \psi$ is constant (and there exists a multi-index $\boldsymbol{\alpha}$ with $\|\boldsymbol{\alpha}\| = d$ such that $\partial^{\boldsymbol{\alpha}} \psi$ is nonzero).

3 If $r > d$, then $\partial^{\boldsymbol{\alpha}} \psi \equiv 0$.

Proof: Writing $\psi(\boldsymbol{z}) = \sum_{|\boldsymbol{k}|=d} \boldsymbol{a}_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}$ with coefficients $\boldsymbol{a}_{\boldsymbol{k}} \in \mathbb{C}^m$, we compute

$$\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{z}) = \sum_{|\boldsymbol{k}|=d} \boldsymbol{a}_{\boldsymbol{k}} \prod_{j=1}^{n} \frac{k_j!}{(k_j - \alpha_j)!} z_j^{k_j - \alpha_j}, \quad \boldsymbol{k} = (k_1, ..., k_n),$$

where terms with $k_j < \alpha_j$ vanish. For each remaining term, the total degree is

$$(k_1 - \alpha_1) + \cdots + (k_n - \alpha_n) = d - \|\boldsymbol{\alpha}\|.$$

Hence, $\partial^{\boldsymbol{\alpha}} \psi$ is a homogeneous polynomial of degree $d - \|\boldsymbol{\alpha}\|$, establishing Part 1.

If $r = d$, every surviving monomial has degree 0, so $\partial^{\alpha}\psi$ is constant. Moreover, since ψ has degree exactly d, there exists some multi-index $\boldsymbol{k}$ with $|\boldsymbol{k}| = d$ and $\boldsymbol{a}_{\boldsymbol{k}} \neq \boldsymbol{0}$; choosing $\boldsymbol{\alpha} = \boldsymbol{k}$ yields a nonzero constant derivative. This proves Part 2.

Finally, if $r > d$, then for every term in the expansion, at least one $k_j < \alpha_j$, so all summands vanish identically. Thus $\partial^{\alpha}\psi \equiv 0$, verifying Part 3. ☐

Lemma 9.2.1 *(Cartan)*: Let $\Omega \subset \mathbb{C}^n$ be a bounded region, and suppose that $\boldsymbol{f} = (f_1, ..., f_n) : \Omega \to \Omega$ is holomorphic. If $\exists \boldsymbol{a} \in \Omega$ such that $\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{a}$ and the complex Jacobian at $\boldsymbol{a}$ is the identity matrix, or equivalently, if

$$\boldsymbol{J}_{\boldsymbol{f}}(\boldsymbol{a}) = \begin{pmatrix} \frac{\partial f_1}{\partial z_1}(\boldsymbol{a}) & \cdots & \frac{\partial f_1}{\partial z_n}(\boldsymbol{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1}(\boldsymbol{a}) & \cdots & \frac{\partial f_n}{\partial z_n}(\boldsymbol{a}) \end{pmatrix} = \boldsymbol{I} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \qquad (9.2.1)$$

then $\boldsymbol{f}(\boldsymbol{z}) \equiv \boldsymbol{z}$ is the identity map.

Proof: By Theorem 9.1.2, we have the expansion

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{z}) &= \sum_{|\boldsymbol{k}|=0}^{\infty} \boldsymbol{a}_{\boldsymbol{k}}(\boldsymbol{z}-\boldsymbol{a})^{\boldsymbol{k}} = \sum_{j=0}^{\infty} \psi_j(\boldsymbol{z}-\boldsymbol{a}) \\ &= \boldsymbol{a} + \sum_{j=1}^{\infty} \sum_{|\boldsymbol{k}|=j} \boldsymbol{a}_{\boldsymbol{k}}(\boldsymbol{z}-\boldsymbol{a})^{\boldsymbol{k}}, \end{aligned} \qquad (9.2.2)$$

which is absolutely convergent on some polydisk centered at $\boldsymbol{a}$, where $\boldsymbol{a}_{\boldsymbol{k}} = \frac{\partial^{\boldsymbol{k}} \boldsymbol{f}(\boldsymbol{a})}{\prod_{j=1}^{n} k_j!}$ and $\boldsymbol{k} = (k_1, ..., k_n)$. The terms have been rearranged (from absolute convergence) so that the inner summation is a homogeneous polynomial ψ_j with a zero at $\boldsymbol{z} = \boldsymbol{a}$ and degree j.

Trivially, $\boldsymbol{a}_{1,0,...,0} = \frac{\partial \boldsymbol{f}}{\partial z_1}(\boldsymbol{a}) = (1, 0, ..., 0)$ by (9.2.1). Similarly, $\boldsymbol{a}_{0,1,0,...,0} = (0, 1, 0, ..., 0), ..., \boldsymbol{a}_{0,...,0,1} = (0, ..., 0, 1)$. Hence, the linear homogeneous polynomial of (9.2.2) equals

$$(z_1 - a_1, ..., z_n - a_n) = \boldsymbol{z} - \boldsymbol{a},$$

and the entire expansion is thus equal to

$$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{z} + \sum_{j=2}^{\infty} \sum_{|\boldsymbol{k}|=j} \boldsymbol{a}_{\boldsymbol{k}}(\boldsymbol{z}-\boldsymbol{a})^{\boldsymbol{k}}.$$

Define a sequence of holomorphic functions $\{\boldsymbol{f}_k(\boldsymbol{z})\}_{k \in \mathbb{N}}$ by

$$\boldsymbol{f}_1 = \boldsymbol{f}, \quad \boldsymbol{f}_{k+1} = \boldsymbol{f}_k \circ \boldsymbol{f} \quad \forall k \in \mathbb{N}.$$

Assume the existence of some $m \in \mathbb{N}$, the smallest $j \geq 2$ such that $\boldsymbol{\psi}$ is not identically zero. Because

$$\boldsymbol{f}_1(z) = \boldsymbol{z} + \boldsymbol{\psi}_m(\boldsymbol{z} - \boldsymbol{a}) + \sum_{j>m} \boldsymbol{\psi}_j(\boldsymbol{z} - \boldsymbol{a}),$$

it then follows that

$$\begin{aligned}
\boldsymbol{f}_2(\boldsymbol{z}) &= \boldsymbol{z} + \boldsymbol{\psi}_m(\boldsymbol{z} - \boldsymbol{a}) + \sum_{j>m} \boldsymbol{\psi}_j(\boldsymbol{z} - \boldsymbol{a}) \\
&\quad + \boldsymbol{\psi}_m\left(\boldsymbol{z} - \boldsymbol{a} + \sum_{j\geq m} \boldsymbol{\psi}_j(\boldsymbol{z} - \boldsymbol{a})\right) + \sum_{j>m} \boldsymbol{\psi}_j(\boldsymbol{f}(\boldsymbol{z}) - \boldsymbol{a}) \\
&= \boldsymbol{z} + 2\boldsymbol{\psi}_m(\boldsymbol{z} - \boldsymbol{a}) \\
&\quad + (\text{homogeneous polynomials of degree} > m)(\boldsymbol{z} - \boldsymbol{a}).
\end{aligned}$$

Assume, for induction, that

$$\begin{aligned}
\boldsymbol{f}_k(\boldsymbol{z}) &= \boldsymbol{z} + k\boldsymbol{\psi}_m(\boldsymbol{z} - \boldsymbol{a}) \\
&\quad + (\text{homogeneous polynomials of degree} > m)(\boldsymbol{z} - \boldsymbol{a}).
\end{aligned}$$

Then we have

$$\begin{aligned}
\boldsymbol{f}_{k+1}(\boldsymbol{z}) &= \boldsymbol{z} + \sum_{j\geq m} \boldsymbol{\psi}_j(\boldsymbol{z} - \boldsymbol{a}) \\
&\quad + k\boldsymbol{\psi}_m\left(\boldsymbol{z} - \boldsymbol{a} + \sum_{j\geq m}(\text{degree } j \text{ hom. polynomial})(\boldsymbol{z} - \boldsymbol{a})\right) \\
&\quad + \sum_{j>m}(\text{homogeneous polynomial of degree } j)(\boldsymbol{f}(\boldsymbol{z}) - \boldsymbol{a}) \\
&= \boldsymbol{z} + (k+1)\boldsymbol{\psi}_m(\boldsymbol{z} - \boldsymbol{a}) \\
&\quad + (\text{degree} > m \text{ homogeneous polynomials})(\boldsymbol{z} - \boldsymbol{a}).
\end{aligned}$$

Since $\boldsymbol{f}_k(\Omega) \subseteq \Omega$ for any k, the sequence $\{\boldsymbol{f}_k\}_{k\in\mathbb{N}}$ is uniformly bounded on Ω. By Montel's Theorem (Theorem 9.1.6), there exists a subsequence $\left\{\boldsymbol{f}_{k_l}\right\}_{l\in\mathbb{N}}$ that converges locally uniformly to some holomorphic function $\tilde{\boldsymbol{f}}$ by virtue of Weierstrass (Theorem 9.1.5).

Since $\boldsymbol{\psi}_m \not\equiv 0$, there exists $\boldsymbol{\alpha}$ satisfying $\|\boldsymbol{\alpha}\| = m$ such that

$$\partial^{\boldsymbol{\alpha}} \boldsymbol{\psi}_m \equiv \boldsymbol{c} \neq \boldsymbol{0}$$

is a nonzero constant by Proposition 9.2.1. Consequently,

$$\partial^{\boldsymbol{\alpha}}(\text{homogeneous polynomials of degree} > m)(\boldsymbol{z} - \boldsymbol{a})$$

is a homogeneous polynomial with degree ≥ 1 and thus vanishes as $\boldsymbol{z} \to \boldsymbol{a}$. Similarly, $\boldsymbol{z} \mapsto \boldsymbol{z}$ is homogeneous with degree $1 < m$ and thus $\partial^{\boldsymbol{\alpha}} z$ vanishes. Therefore,

$$\partial^{\boldsymbol{\alpha}} \boldsymbol{f}_k(\boldsymbol{a}) = k\boldsymbol{c},$$

which diverges as $k \to \infty$. Weierstrass' Convergence Theorem (Theorem 9.1.5) gives that $\partial^{\boldsymbol{\alpha}} \boldsymbol{f}_{k_l}(\boldsymbol{a}) \to \partial^{\boldsymbol{\alpha}} \tilde{\boldsymbol{f}}(\boldsymbol{a})$ which must be finite by holomorphy, contradicting the divergence. Hence, the assumed value for m cannot exist and hence $\psi_j \equiv 0$ for all $j \geq 2$. Thus, $\boldsymbol{f}(\boldsymbol{z}) \equiv \boldsymbol{z}$ on some polydisk centered at $\boldsymbol{a}$. By the Identity Theorem (Theorem 9.1.3), $\boldsymbol{f}(\boldsymbol{z}) \equiv \boldsymbol{z}$ on Ω. $\square$

Definition 9.2.3 *(Reinhardt Domain)*: An open domain $\Omega \subseteq \mathbb{C}^n$ is a *Reinhardt domain* centered at $\boldsymbol{a} = (a_1, ..., a_n) \in \mathbb{C}^n$ iff $\forall \boldsymbol{\zeta} = (\zeta_1, ..., \zeta_n) \in \Omega$,

$$\{(z_1, ..., z_n) \in \mathbb{C}^n : |z_k - a_k| = |\zeta_k - a_k|, 1 \leq k \leq n\}$$

is fully contained in Ω. In other words, Ω is invariant under all rotations about the center $\boldsymbol{a}$ in each coordinate.

Definition 9.2.4: A Reinhardt domain $\Omega \subseteq \mathbb{C}^n$ centered at $\boldsymbol{a} = (a_1, ..., a_n)$ is said to be *complete* iff $\forall \boldsymbol{\zeta} = (\zeta_1, ..., \zeta_n) \in \Omega$, the polydisk

$$\{(z_1, ..., z_n) \in \mathbb{C}^n : |z_k - a_k| \leq |\zeta_k - a_k|, 1 \leq k \leq n\}$$

is contained in Ω.

Definition 9.2.5 *(Circular Domain)*: An open domain $\Omega \subseteq \mathbb{C}^n$ is a *circular domain* centered at $\boldsymbol{a} \in \mathbb{C}^n$ iff $\forall \boldsymbol{\zeta} \in \Omega$,

$$\left\{\boldsymbol{a} + \mathrm{e}^{\mathrm{i}\theta}(\boldsymbol{\zeta} - \boldsymbol{a}) : 0 \leq \theta < 2\pi\right\}$$

is fully contained in Ω.

Definition 9.2.6: A circular domain $\Omega \subseteq \mathbb{C}^n$ centered at $\boldsymbol{a} = (a_1, ..., a_n)$ is said to be *complete* iff $\forall \boldsymbol{\zeta} \in \Omega$,

$$\left\{\boldsymbol{a} + \mu(\boldsymbol{\zeta} - \boldsymbol{a}) : \forall \mu \in \overline{\mathbb{D}}\right\}$$

is contained in Ω.

Proposition 9.2.2: Let $U_0 \subseteq \mathbb{C}^{n_0}, U_1 \subseteq \mathbb{C}^{n_1}, U_2 \subseteq \mathbb{C}^{n_2}$ be open domains with $n_i \geq 1$ for each i, and let $\boldsymbol{f} : U_1 \to U_2$ and $\boldsymbol{g} : U_0 \to U_1$ be holomorphic maps. Define the composition $\boldsymbol{h} : U_0 \to U_2$ by $\boldsymbol{h}(\boldsymbol{z}) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{z}))$. Then for every $\boldsymbol{z} \in U_0$, the complex Jacobian matrix of $\boldsymbol{h}$ at $\boldsymbol{z}$ is

$$\boldsymbol{J}_{\boldsymbol{h}}(\boldsymbol{z}) = \boldsymbol{J}_{\boldsymbol{f}}(\boldsymbol{g}(\boldsymbol{z})) \cdot \boldsymbol{J}_{\boldsymbol{g}}(\boldsymbol{z}).$$

Proof: Fix $\boldsymbol{z} \in U_0$ and let $\boldsymbol{w} = \boldsymbol{g}(\boldsymbol{z}) \in U_1$. Write

$$\boldsymbol{h}(\boldsymbol{z}) = \left(h_1(\boldsymbol{z}), ..., h_{n_2}(\boldsymbol{z})\right),$$

where each $h_l : U_0 \to \mathbb{C}$ is holomorphic for $l = 1, ..., n_2$. Similarly, write

$$\boldsymbol{g}(\boldsymbol{z}) = \left(g_1(\boldsymbol{z}), ..., g_{n_1}(\boldsymbol{z})\right), \quad \boldsymbol{f}(\boldsymbol{z}) = \left(f_1(\boldsymbol{z}), ..., f_{n_2}(\boldsymbol{z})\right),$$

where each $g_p : U_0 \to \mathbb{C}$ and each $f_l : U_1 \to \mathbb{C}$ is holomorphic for $p = 1, ..., n_1$ and $l = 1, ..., n_2$. Then $h_l(\boldsymbol{z}) = f_l(\boldsymbol{g}(\boldsymbol{z}))$ for each l. By the chain multivariable rule, the complex Jacobian of $\boldsymbol{h}$ at $\boldsymbol{z}$ is the $n_2 \times n_0$ matrix

$$\begin{aligned}
\boldsymbol{J}_{\boldsymbol{h}} &= \begin{pmatrix} \frac{\partial h_1}{\partial z_1} & ... & \frac{\partial h_1}{\partial z_{n_0}} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_{n_2}}{\partial z_1} & ... & \frac{\partial h_{n_2}}{\partial z_{n_0}} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{p=1}^{n_1} \frac{\partial f_1}{\partial g_p}(\boldsymbol{g}) \frac{\partial g_p}{\partial z_1} & ... & \sum_{p=1}^{n_1} \frac{\partial f_1}{\partial g_p}(\boldsymbol{g}) \frac{\partial g_p}{\partial z_{n_0}} \\ \vdots & \ddots & \vdots \\ \sum_{p=1}^{n_1} \frac{\partial f_{n_2}}{\partial g_p}(\boldsymbol{g}) \frac{\partial g_p}{\partial z_1} & ... & \sum_{p=1}^{n_1} \frac{\partial f_{n_2}}{\partial g_p}(\boldsymbol{g}) \frac{\partial g_p}{\partial z_{n_0}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial f_1}{\partial g_1}(\boldsymbol{g}) & ... & \frac{\partial f_1}{\partial g_{n_1}}(\boldsymbol{g}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{n_2}}{\partial g_1}(\boldsymbol{g}) & ... & \frac{\partial f_{n_2}}{\partial g_{n_1}}(\boldsymbol{g}) \end{pmatrix} \begin{pmatrix} \frac{\partial g_1}{\partial z_1} & ... & \frac{\partial g_1}{\partial z_{n_0}} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{n_1}}{\partial z_1} & ... & \frac{\partial g_{n_1}}{\partial z_{n_0}} \end{pmatrix} = \boldsymbol{J}_{\boldsymbol{f}}(\boldsymbol{g}) \cdot \boldsymbol{J}_{\boldsymbol{g}}.
\end{aligned}$$

□

Lemma 9.2.2 *(Cartan)*: Let $\Omega \subset \mathbb{C}^n$ be a bounded complete circular domain centered at $\mathbf{0}$, and suppose that $\boldsymbol{f} = (f_1, ..., f_n) : \Omega \to \Omega$ is a biholomorphism. If $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$, then $\boldsymbol{f}$ is linear.

Proof: Let $\boldsymbol{\rho}_\theta(\boldsymbol{z}) = \mathrm{e}^{\mathrm{i}\theta}\boldsymbol{z}$ for all $\theta \in \mathbb{R}$ and suppose that $\boldsymbol{\varphi} = \boldsymbol{\rho}_{-\theta} \circ \boldsymbol{f}^{-1} \circ \boldsymbol{\rho}_\theta \circ \boldsymbol{f}$. By Proposition 9.2.2, we must have that

$$\begin{aligned}
\boldsymbol{J}_{\boldsymbol{\varphi}}(\boldsymbol{z}) &= \boldsymbol{J}_{\boldsymbol{\rho}_{-\theta}}(\boldsymbol{f}^{-1} \circ \boldsymbol{\rho}_\theta \circ \boldsymbol{f}(\boldsymbol{z})) \cdot \boldsymbol{J}_{\boldsymbol{\rho}_{-\theta} \circ \boldsymbol{f}^{-1}}(\boldsymbol{\rho}_\theta \circ \boldsymbol{f}(\boldsymbol{z})) \\
&\quad \cdot \boldsymbol{J}_{\boldsymbol{\rho}_{-\theta} \circ \boldsymbol{f}^{-1} \circ \boldsymbol{\rho}_\theta}(\boldsymbol{f}(\boldsymbol{z})) \cdot \boldsymbol{J}_{\boldsymbol{\rho}_{-\theta} \circ \boldsymbol{f}^{-1} \circ \boldsymbol{\rho}_\theta \circ \boldsymbol{f}}(\boldsymbol{z}) \\
\boldsymbol{J}_{\boldsymbol{\varphi}}(\mathbf{0}) &= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & ... & 0 \\ \vdots & \ddots & \vdots \\ 0 & ... & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix} \cdot \boldsymbol{J}_{\boldsymbol{f}^{-1}}(\mathbf{0}) \cdot \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & ... & 0 \\ \vdots & \ddots & \vdots \\ 0 & ... & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \cdot \boldsymbol{J}_{\boldsymbol{f}}(\mathbf{0}) \\
&= \mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\theta}\left(\boldsymbol{J}_{\boldsymbol{f}^{-1}} \cdot \boldsymbol{J}_{\boldsymbol{f}}\right)(\mathbf{0}) = \boldsymbol{I}.
\end{aligned}$$

By Lemma 9.2.1, $\boldsymbol{\varphi}(\boldsymbol{z}) \equiv \boldsymbol{z}$ on Ω. Hence, $\boldsymbol{f} \circ \boldsymbol{\rho}_\theta = \boldsymbol{\rho}_\theta \circ \boldsymbol{f}$ for all $\theta \in \mathbb{R}$. Together with Theorem 9.1.2, write

$$\boldsymbol{f}(\boldsymbol{z}) = \sum_{\boldsymbol{k}:|\boldsymbol{k}|=0}^{\infty} \boldsymbol{a}_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}} \tag{9.2.3}$$

on a polydisk centered at $\mathbf{0}$. Thus,

$$\boldsymbol{f} \circ \boldsymbol{\rho}(\boldsymbol{z}) = \sum_{\boldsymbol{k}:|\boldsymbol{k}|=0}^{\infty} \boldsymbol{a}_{\boldsymbol{k}} \left(\mathrm{e}^{\mathrm{i}\theta} \boldsymbol{z}\right)^{\boldsymbol{k}} = \sum_{\boldsymbol{k}:|\boldsymbol{k}|=0}^{\infty} \boldsymbol{a}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\theta|\boldsymbol{k}|} \boldsymbol{z}^{\boldsymbol{k}}.$$

On the other hand, composing with $\boldsymbol{\rho}_\theta$ with (9.2.3) gives

$$\boldsymbol{\rho}_\theta \circ \boldsymbol{f}(\boldsymbol{z}) = \mathrm{e}^{\mathrm{i}\theta} \sum_{\boldsymbol{k}:|\boldsymbol{k}|=0}^{\infty} \boldsymbol{a}_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}} = \sum_{\boldsymbol{k}:|\boldsymbol{k}|=0}^{\infty} \boldsymbol{a}_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\theta} \boldsymbol{z}^{\boldsymbol{k}}.$$

Hence, by the uniqueness of power series expansions, we must either have that $\boldsymbol{a}_{\boldsymbol{k}} = \mathbf{0}$, $\mathrm{e}^{\mathrm{i}\theta} \equiv \mathrm{e}^{\mathrm{i}\theta|\boldsymbol{k}|}$, or equivalently, that $\forall \theta \in \mathbb{R}$,

$$\theta|\boldsymbol{k}| \equiv \theta \bmod 2\pi \Longrightarrow |\boldsymbol{k}| \equiv 1 \bmod 2\pi \text{ (by letting } \theta = 1\text{)}.$$

This is only possible when $|\boldsymbol{k}| = 1$ by irrationality, and thus $\boldsymbol{a}_{\boldsymbol{k}} = \mathbf{0}$ for all $|\boldsymbol{k}| \neq 1$. Therefore, $\boldsymbol{f}$ must be linear. □

Remark: If $n = 1$, then $\Omega = D(0, R)$ for some $R > 0$ and any automorphism f with a fixed point 0 is a rotation in the form of $z \mapsto \mathrm{e}^{\mathrm{i}\theta} z$, hence linear, the effective statement of the Schwarz Lemma (Lemma 3.5.1).

Theorem 9.2.1 *(THE HOLOMORPHIC AUTOMORPHISM GROUP ON $\mathbb{D}^n$)*: The holomorphic automorphism group of the polydisk $\mathbb{D}^n$ consists solely of biholomorphisms in the form of

$$\boldsymbol{z} = (z_1, ..., z_n) \mapsto \boldsymbol{P}\left(\mathrm{e}^{\mathrm{i}\theta_1} \frac{z_1 - a_1}{1 - \overline{a_1} z_1}, ..., \mathrm{e}^{\mathrm{i}\theta_n} \frac{z_n - a_n}{1 - \overline{a_n} z_n}\right), \tag{9.2.4}$$

where $\boldsymbol{P}$ is a $n \times n$ permutation matrix (for coordinate permutations), $(\theta_1, ..., \theta_n) \in \mathbb{R}^n$, and $(a_1, ..., a_n) \in \mathbb{D}^n$. Moreover, every such map is indeed an automorphism.

Proof: Let $\boldsymbol{f} \in \mathrm{Aut}(\mathbb{D}^n)$ be arbitrary, and set $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) = \boldsymbol{f}(\mathbf{0})$. Define the Möbius transformation $\boldsymbol{\varphi}(z_1, ..., z_n) = \left(\frac{z_1 - \alpha_1}{1 - \overline{\alpha_1} z_1}, ..., \frac{z_n - \alpha_n}{1 - \overline{\alpha_n} z_n}\right) \in \mathrm{Aut}(\mathbb{D}^n)$. It follows that $\boldsymbol{\varphi} \circ \boldsymbol{f}(\mathbf{0}) = \mathbf{0}$ and $\boldsymbol{\varphi} \circ \boldsymbol{f} \in \mathrm{Aut}(\mathbb{D}^n)$.

By Lemma 9.2.2, the map $\boldsymbol{\varphi} \circ \boldsymbol{f}$ is linear, so $\boldsymbol{\varphi} \circ \boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{A}\boldsymbol{z}$ for some invertible constant matrix $\boldsymbol{A} = \begin{pmatrix} \zeta_{1,1} & \cdots & \zeta_{1,n} \\ \vdots & \ddots & \vdots \\ \zeta_{n,1} & \cdots & \zeta_{n,n} \end{pmatrix}$, hence $\boldsymbol{A} \in \mathrm{Aut}(\mathbb{D}^n)$. Thus,

$$\left|\sum_{j=1}^{n} \zeta_{k,j} z_j\right| < 1 \quad \forall \boldsymbol{z} \in \mathbb{D}^n, \forall k \in \{1, ..., n\},$$

which implies $\left|\zeta_{k,j}\right| \leq 1$ for all $j, k \in \{1, ..., n\}$ (for if $\left|\zeta_{k,j}\right| > 1$, then choosing $z_j = \frac{1}{\left|\zeta_{k,j}\right|} + \varepsilon$ with $0 < \varepsilon < 1 - \frac{1}{\left|\zeta_{k,j}\right|}$ and $z_l = 0$ for $l \neq j$ yields a contradiction).

For each $j \in \{1, ..., n\}$, define the sequence $\left\{\boldsymbol{z}_{j,k}\right\}_{k \in \mathbb{N}}$ for each $k \in \mathbb{N}$ by

$$\boldsymbol{z}_{j,k} = \left(z_{j,k,1}, ..., z_{j,k,n}\right) = \left(\left(1 - \frac{1}{k}\right)\frac{\left|\zeta_{j,1}\right|}{\zeta_{j,1}}, ..., \left(1 - \frac{1}{k}\right)\frac{\left|\zeta_{j,n}\right|}{\zeta_{j,n}}\right) \in \mathbb{D}^n,$$

where we informally let $\frac{\left|\zeta_{j,i}\right|}{\zeta_{j,i}} = 0$ if $\zeta_{j,i} = 0$. Then, for all $j \in \{1, ..., n\}$ and $k \in \mathbb{N}$,

$$\boldsymbol{\varphi} \circ \boldsymbol{f}\left(\boldsymbol{z}_{j,k}\right) = \left(1 - \frac{1}{k}\right)\begin{pmatrix} \sum_{i=1}^{n}\left(\frac{\left|\zeta_{j,i}\right|}{\zeta_{j,i}}\right)\zeta_{1,i} \\ \vdots \\ \sum_{i=1}^{n}\left(\frac{\left|\zeta_{j,i}\right|}{\zeta_{j,i}}\right)\zeta_{j,i} \\ \vdots \\ \sum_{i=1}^{n}\left(\frac{\left|\zeta_{j,i}\right|}{\zeta_{j,i}}\right)\zeta_{n,i} \end{pmatrix} \in \mathbb{D}^n.$$

In particular, the j-th component is

$$\left(1 - \frac{1}{k}\right)\sum_{i=1}^{n}\left|\zeta_{j,i}\right| \in \mathbb{D}.$$

As $k \to \infty$,

$$\sum_{i=1}^{n}\left|\zeta_{j,i}\right| \leq 1 \qquad \forall j \in \{1, ..., n\}. \tag{9.2.5}$$

Now consider, for each $j \in \{1, ..., n\}$, the sequence $\boldsymbol{z}'_{j,k} = \left(0, ..., 0, 1 - \frac{1}{k}, 0, ..., 0\right)$, where $1 - \frac{1}{k}$ is in the j-th position. Then

$$\boldsymbol{\varphi} \circ \boldsymbol{f}\left(\boldsymbol{z}'_{j,k}\right) = \left(1 - \frac{1}{k}\right)\left(\zeta_{1,j}, ..., \zeta_{n,j}\right).$$

As $k \to \infty$, $\boldsymbol{z}'_{j,k} \to \boldsymbol{e}_j \in \partial(\mathbb{D}^n)$ (the j-th unit basis vector), so the limit is

$$\boldsymbol{\zeta}_j = \left(\zeta_{1,j}, ..., \zeta_{n,j}\right) \in \overline{\mathbb{D}^n}.$$

Because the function $\boldsymbol{A}$ is injective on all of $\mathbb{C}^n$, if $\boldsymbol{\zeta}_j \in \mathbb{D}^n$ (within the interior), then $\boldsymbol{A}^{-1} \in \operatorname{Aut}(\mathbb{D}^n)$ would map $\boldsymbol{\zeta}_j$ to $\boldsymbol{e}_j \in \partial(\mathbb{D}^n)$, which is an impossibility. Hence, $\boldsymbol{\zeta}_j \in \partial(\mathbb{D}^n)$, and consequently, $\max_{i \in \{1,...,n\}} \left|\zeta_{i,j}\right| = 1$. Combined with (9.2.5), this forces exactly one entry in the j-th column of $\boldsymbol{A}$ to have absolute value 1 (of the form $\mathrm{e}^{\mathrm{i}\theta_j}$), with all others zero.

Invertibility of $\boldsymbol{A}$ ensures each column has at least one nonzero entry, so $\boldsymbol{A}$ is a monomial matrix, which factors to

$$\boldsymbol{A} = \boldsymbol{P} \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\theta_1}, ..., \mathrm{e}^{\mathrm{i}\theta_n}\right)$$

for some permutation matrix $\boldsymbol{P}$. Therefore,

$$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{\varphi}^{-1} \circ \left(\boldsymbol{P} \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\theta_1}, ..., \mathrm{e}^{\mathrm{i}\theta_n}\right) \boldsymbol{z}\right).$$

Let $\sigma : \mathbb{N}_{\leq n} \to \mathbb{N}_{\leq n}$ be the permutation induced by $\boldsymbol{P}$. The map $\boldsymbol{A}$ multiplies the m-th input coordinate z_m by $\mathrm{e}^{\mathrm{i}\theta_m}$ and permutes to place it in the $\sigma(m)$-th output position, so the $\sigma(m)$-th coordinate of $\boldsymbol{A}\boldsymbol{z}$ is $\mathrm{e}^{\mathrm{i}\theta_m} z_m$. Applying $\boldsymbol{\varphi}^{-1}$ componentwise then gives, for the k-th output coordinate,

$$(\boldsymbol{f}(\boldsymbol{z}))_k = \varphi_{\alpha_k}^{-1}\left(\mathrm{e}^{\mathrm{i}\theta_{\sigma^{-1}(k)}} z_{\sigma^{-1}(k)}\right) = \frac{\mathrm{e}^{\mathrm{i}\theta_{\sigma^{-1}(k)}} z_{\sigma^{-1}(k)} + \alpha_k}{1 + \overline{\alpha_k} \mathrm{e}^{\mathrm{i}\theta_{\sigma^{-1}(k)}} z_{\sigma^{-1}(k)}}.$$

Set $a_{\sigma^{-1}(k)} = -\alpha_k \mathrm{e}^{-\mathrm{i}\theta_{\sigma^{-1}(k)}} \in \mathbb{D}$. Then

$$(\boldsymbol{f}(\boldsymbol{z}))_k = \frac{\mathrm{e}^{\mathrm{i}\theta_{\sigma^{-1}(k)}} z_{\sigma^{-1}(k)} + \alpha_k}{1 + \overline{\alpha_k} \mathrm{e}^{\mathrm{i}\theta_{\sigma^{-1}(k)}} z_{\sigma^{-1}(k)}} = \mathrm{e}^{\mathrm{i}\theta_{\sigma^{-1}(k)}} \frac{z_{\sigma^{-1}(k)} - a_{\sigma^{-1}(k)}}{1 - \overline{a_{\sigma^{-1}(k)}} z_{\sigma^{-1}(k)}}.$$

Hence,

$$(\boldsymbol{f}(\boldsymbol{z}))_{\sigma(k)} = \mathrm{e}^{\mathrm{i}\theta_k} \frac{z_k - a_k}{1 - \overline{a_k} z_k} \Longleftrightarrow \boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{P}\left(\mathrm{e}^{\mathrm{i}\theta_1} \frac{z_1 - a_1}{1 - \overline{a_1} z_1}, ..., \mathrm{e}^{\mathrm{i}\theta_n} \frac{z_n - a_n}{1 - \overline{a_n} z_n}\right),$$

as in (9.2.4). Finally, each automorphism of this form lies in $\operatorname{Aut}(\mathbb{D}^n)$ trivially. □

Definition 9.2.7: The *conjugate transpose* or *Hermitian transpose* of a complex matrix $\boldsymbol{U}$ is defined as $\boldsymbol{U}^\dagger = \overline{\boldsymbol{U}}^\top$, or the transpose of the matrix with each element replaced with its complex conjugate.

Definition 9.2.8: A matrix $\boldsymbol{U}$ is said to be *unitary* iff its inverse is its conjugate transpose, or iff $\boldsymbol{U}^\dagger \boldsymbol{U} = \boldsymbol{U}\boldsymbol{U}^\dagger = \boldsymbol{I}$.

Definition 9.2.9: A matrix $\boldsymbol{M}$ is said to be *monomial* iff it has exactly one nonzero entry in each row and each column.

Theorem 9.2.2 *(Spectral Theorem)*: For any unitary matrix $\boldsymbol{U}$, there exists a unitary matrix $\boldsymbol{V}$ such that $\boldsymbol{U} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^\dagger$, where $\boldsymbol{D}$ is a diagonal matrix whose diagonal entries are all of unit modulus.

Proof: Because $\|\boldsymbol{U}\boldsymbol{z}\|^2 = \boldsymbol{z}^\dagger\boldsymbol{U}^\dagger\boldsymbol{U}\boldsymbol{z} = \|\boldsymbol{z}\|^2$ for any $\boldsymbol{z} \in \mathbb{C}^n$, any eigenvalue λ_1 (existence given by the Fundamental Theorem of Algebra in Theorem 3.3.1 on the characteristic equation) of $\boldsymbol{U}$ must satisfy

$$\boldsymbol{U}\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_1 \Longrightarrow \|\boldsymbol{U}\boldsymbol{v}_1\| = \|\boldsymbol{v}_1\| = |\lambda_1|\|\boldsymbol{v}_1\| \Longrightarrow |\lambda_1| = 1,$$

where $\|\boldsymbol{v}_1\| = 1$ is the corresponding eigenvector in $\mathbb{C}^n$. Then

$$\boldsymbol{U}^{-1}\boldsymbol{U}\boldsymbol{v}_1 = \boldsymbol{U}^{-1}\lambda_1\boldsymbol{v}_1 \Longrightarrow \frac{1}{\lambda_1}\boldsymbol{v}_1 = \boldsymbol{U}^{-1}\boldsymbol{v}_1 \Longrightarrow \overline{\lambda_1}\boldsymbol{v}_1 = \boldsymbol{U}^\dagger\boldsymbol{v}_1.$$

Let $\boldsymbol{v}_1^\perp = \left\{\boldsymbol{w} : \boldsymbol{v}_1^\dagger\boldsymbol{w} = \boldsymbol{0}\right\} \subset \mathbb{C}^n$ be an $(n-1)$-dimensional subspace. For any $\boldsymbol{w} \in \boldsymbol{v}_1^\perp$,

$$\boldsymbol{v}_1^\dagger\boldsymbol{U}\boldsymbol{w} = \left(\boldsymbol{U}^\dagger\boldsymbol{v}_1\right)^\dagger\boldsymbol{w} = \left(\overline{\lambda_1}\boldsymbol{v}_1\right)^\dagger\boldsymbol{w} = \lambda_1\boldsymbol{v}_1^\dagger\boldsymbol{w} = 0,$$

so $\boldsymbol{U}\boldsymbol{w} \in \boldsymbol{v}_1^\perp$. Hence $\boldsymbol{v}_1^\perp$ is invariant under $\boldsymbol{U}$. The restriction of $\boldsymbol{U}$ to $\boldsymbol{v}_1^\perp$, $\boldsymbol{U}|_{\boldsymbol{v}_1^\perp}$, yields another eigenvalue $\lambda_2 \in \partial\mathbb{D}$ with eigenvector $\boldsymbol{v}_2 \in \boldsymbol{v}_1^\perp$ satisfying $|\lambda_2| = 1$ and $\|\boldsymbol{v}_2\| = 1$. Similarly, we may define $\boldsymbol{v}_2^\perp \subset \boldsymbol{v}_1^\perp$, which is an $(n-2)$-dimensional subspace invariant under $\boldsymbol{U}$. Repeating this process inductively, we obtain an orthonormal basis $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\}$ of eigenvectors of $\boldsymbol{U}$ with corresponding eigenvalues $\lambda_1, ..., \lambda_n \in \partial\mathbb{D}$. Setting

$$\boldsymbol{V} = \begin{pmatrix}\boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n\end{pmatrix}, \quad \boldsymbol{D} = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$$

gives that

$$\boldsymbol{V}^\dagger\boldsymbol{U}\boldsymbol{V} = \boldsymbol{V}^\dagger\begin{pmatrix}\boldsymbol{U}\boldsymbol{v}_1 & \cdots & \boldsymbol{U}\boldsymbol{v}_n\end{pmatrix} = \boldsymbol{V}^\dagger\begin{pmatrix}\lambda_1\boldsymbol{v}_1 & \cdots & \lambda_n\boldsymbol{v}_n\end{pmatrix} = \boldsymbol{V}^\dagger\boldsymbol{V}\boldsymbol{D}.$$

The k-th diagonal entry of $\boldsymbol{V}^\dagger\boldsymbol{V}$ is equal to $\boldsymbol{v}_k^\dagger\boldsymbol{v}_k = \|\boldsymbol{v}_k\|^2 = 1$, while the non-diagonal entries correspond to $\boldsymbol{v}_k^\dagger\boldsymbol{v}_l$ for some $k \neq l$, which vanish by orthogonality in construction. Thus, $\boldsymbol{V}^\dagger\boldsymbol{V} = \boldsymbol{I}$ (unitary) and $\boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^\dagger = \boldsymbol{U}$.□

A *unitary transformation* is a map in the form of $\boldsymbol{z} \mapsto \boldsymbol{U}\boldsymbol{z}$, where $\boldsymbol{U}$ is a unitary matrix.

Proposition 9.2.3: For any $a \in \mathbb{D}$,

$$\begin{aligned}\boldsymbol{w} = (w_1, ..., w_n) &= \boldsymbol{\varphi}_a(\boldsymbol{z}) \\ &= \left(\frac{z_1 - a}{1 - \overline{a}z_1}, z_2\frac{\sqrt{1-|a|^2}}{1 - \overline{a}z_1}, z_3\frac{\sqrt{1-|a|^2}}{1 - \overline{a}z_1}, ..., z_n\frac{\sqrt{1-|a|^2}}{1 - \overline{a}z_1}\right)\end{aligned} \tag{9.2.6}$$

lies in $\mathrm{Aut}(B^n)$, where $\boldsymbol{z} = (z_1, ..., z_n)$. Moreover, $\boldsymbol{\varphi}_a^{-1} = \boldsymbol{\varphi}_{-a}$.

Proof: For $\boldsymbol{z} = (z_1, ..., z_n) \in B^n$, because $\sum_{k=2}^n |z_k|^2 < 1 - |z_1|^2$,

$$\begin{aligned}\|\boldsymbol{\varphi}_a(\boldsymbol{z})\|^2 &= \frac{1}{|1 - \overline{a}z_1|^2}\left[|z_1 - a|^2 + \sum_{k=2}^n (1 - |a|^2)|z_k|^2\right] \\ &< \frac{1}{(1 - \overline{a}z_1)(1 - a\overline{z_1})}\left[(z_1 - a)(\overline{z_1} - \overline{a}) + \left(1 - |z_1|^2\right)(1 - |a|^2)\right] \\ &= \frac{|z_1|^2 + |a|^2 - 2\,\mathfrak{Re}(\overline{a}z_1) + 1 + |az_1|^2 - |a|^2 - |z_1|^2}{1 + |az_1|^2 - 2\,\mathfrak{Re}(\overline{a}z_1)} = 1.\end{aligned}$$

Hence, $\boldsymbol{\varphi}_a$ maps B^n to B^n. A simple calculation shows that

$$w_1 = \frac{z_1 - a}{1 - \overline{a}z_1} \Longrightarrow z_1 = \frac{w_1 + a}{1 + \overline{a}w_1}, \quad z_k = w_k\frac{1 - \overline{a}z_1}{\sqrt{1-|a|^2}} = w_k\frac{\sqrt{1-|a|^2}}{1 + \overline{a}w_1},$$

and hence $\boldsymbol{\varphi}_a$ is bijective, admitting the inverse $\boldsymbol{\varphi}_{-a}$. Therefore, $\boldsymbol{\varphi}_a \in \mathrm{Aut}(B^n)$. □

Proposition 9.2.4: A function $\boldsymbol{f}$ is a unitary transformation iff $\boldsymbol{f} \in \mathrm{Aut}(B^n)$ and $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$.

Proof: Because B^n is a bounded complete circular domain centered at $\boldsymbol{0}$, from Lemma 9.2.2 we have that $\boldsymbol{f} \equiv \boldsymbol{U}$ for some constant invertible matrix

$$\boldsymbol{U} = \begin{pmatrix} \zeta_{1,1} & \cdots & \zeta_{1,n} \\ \vdots & \ddots & \vdots \\ \zeta_{n,1} & \cdots & \zeta_{n,n}\end{pmatrix}.$$

Similarly, we have $\boldsymbol{f}^{-1} = \boldsymbol{U}^{-1}$, so $\|\boldsymbol{z}\| = \left\|\boldsymbol{U}^{-1}\boldsymbol{U}\boldsymbol{z}\right\|$. Observe that

$$\left\|\frac{1}{\|\boldsymbol{z}\|}\boldsymbol{f}(\boldsymbol{z})\right\| = \left\|\boldsymbol{f}\left(\frac{\boldsymbol{z}}{\|\boldsymbol{z}\|}\right)\right\| = 1 \Longrightarrow \|\boldsymbol{U}\boldsymbol{z}\|^2 = \|\boldsymbol{z}\|^2.$$

More explicitly, we have

$$\boldsymbol{U}\boldsymbol{z} = \left(\sum_{k=1}^n \zeta_{1,k}z_k, ..., \sum_{k=1}^n \zeta_{n,k}z_k\right) \Longrightarrow \|\boldsymbol{U}\boldsymbol{z}\|^2 = \sum_{j=1}^n \left|\sum_{k=1}^n \zeta_{j,k}z_k\right|^2.$$

Letting $\boldsymbol{z} = \boldsymbol{e}_i$ $(1 \le i \le n)$ be the i-th unit basis vector, we obtain

$$\|\boldsymbol{U}\boldsymbol{z}\| = 1 = \left\|\left(\zeta_{1,i}, ..., \zeta_{n,i}\right)\right\|^2 = \sum_{k=1}^{n} \left|\zeta_{k,i}\right|^2 = \sum_{k=1}^{n} \zeta_{k,i}\overline{\zeta_{k,i}}. \tag{9.2.7}$$

Letting $\boldsymbol{z} = \frac{\sqrt{2}}{2}\left(\boldsymbol{e}_i + \boldsymbol{e}_j\right)$ $(i \neq j)$, we have

$$\begin{aligned}\|\boldsymbol{U}\boldsymbol{z}\| = 1 &= \frac{1}{2}\left\|\left(\zeta_{1,i} + \zeta_{1,j}, ..., \zeta_{n,i} + \zeta_{n,j}\right)\right\|^2 = \frac{1}{2}\sum_{k=1}^{n} \left|\zeta_{k,i} + \zeta_{k,j}\right|^2 \\ &= \frac{1}{2}\sum_{k=1}^{n}\left(\left|\zeta_{k,i}^2\right| + \left|\zeta_{k,j}^2\right| + 2\,\mathfrak{Re}\left(\zeta_{k,i}\overline{\zeta_{k,j}}\right)\right) = 1 + \sum_{k=1}^{n} \mathfrak{Re}\left(\zeta_{k,i}\overline{\zeta_{k,j}}\right),\end{aligned}$$

which implies that $\sum_{k=1}^{n} \mathfrak{Re}\left(\zeta_{k,i}\overline{\zeta_{k,j}}\right) = 0$. Similarly, letting $\boldsymbol{z} = \frac{\sqrt{2}}{2}\left(\boldsymbol{e}_i + \mathrm{i}\boldsymbol{e}_j\right)$ gives

$$\begin{aligned}\|\boldsymbol{U}\boldsymbol{z}\| = 1 &= \frac{1}{2}\left\|\left(\zeta_{1,i} + \mathrm{i}\zeta_{1,j}, ..., \zeta_{n,i} + \mathrm{i}\zeta_{n,j}\right)\right\|^2 = \frac{1}{2}\sum_{k=1}^{n} \left|\zeta_{k,i} + \mathrm{i}\zeta_{k,j}\right|^2 \\ &= \frac{1}{2}\sum_{k=1}^{n}\left(\left|\zeta_{k,i}^2\right| + \left|\zeta_{k,j}^2\right| + 2\,\mathfrak{Im}\left(\zeta_{k,i}\overline{\zeta_{k,j}}\right)\right) = 1 + \sum_{k=1}^{n} \mathfrak{Im}\left(\zeta_{k,i}\overline{\zeta_{k,j}}\right),\end{aligned}$$

which implies that $\sum_{k=1}^{n} \mathfrak{Im}\left(\zeta_{k,i}\overline{\zeta_{k,j}}\right) = 0$. Therefore, by (9.2.7), for all $i, j \in \{1, ..., n\}$, observe that

$$\left(\boldsymbol{U}^{\dagger}\boldsymbol{U}\right)_{j,i} = \sum_{k=1}^{n} \zeta_{k,i}\overline{\zeta_{k,j}} = \delta_{j,i},$$

where $\delta_{j,i}$ is the Kronecker delta. Hence, we have $\boldsymbol{U}^{\dagger}\boldsymbol{U} = \boldsymbol{I}$, and thus $\boldsymbol{U}$ is unitary.

Conversely, if $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{U}\boldsymbol{z}$ for some unitary matrix $\boldsymbol{U}$, then for any $\boldsymbol{z} \in B^n$,

$$\|\boldsymbol{f}(\boldsymbol{z})\|^2 = \|\boldsymbol{U}\boldsymbol{z}\|^2 = \boldsymbol{z}^{\dagger}\boldsymbol{U}^{\dagger}\boldsymbol{U}\boldsymbol{z} = \boldsymbol{z}^{\dagger}\boldsymbol{z} = \|\boldsymbol{z}\|^2,$$

so $\boldsymbol{f}$ maps B^n to B^n. Since $\boldsymbol{U}$ is invertible with unitary inverse $\boldsymbol{U}^{\dagger}$, the map $\boldsymbol{f}$ is bijective with inverse $\boldsymbol{f}^{-1}(\boldsymbol{w}) = \boldsymbol{U}^{\dagger}\boldsymbol{w}$, which also maps B^n to B^n. Therefore, $\boldsymbol{f} \in \mathrm{Aut}(B^n)$ and $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. □

Definition 9.2.10: A group G (under juxtaposition) is said to be *divisible* iff for every $g \in G$ and every positive integer n, there exists some $h \in G$ such that $h^n = g$.

Proposition 9.2.5: The divisibility of a group is preserved under group isomorphisms.

Proof: Let $\varphi : G \to H$ be a group isomorphism between groups G and H with juxtaposition.

Assume G is divisible. Fix $y \in H$ and a positive integer n. Since φ is bijective there is $x \in G$ with $\varphi(x) = y$. By divisibility of G there exists $h \in G$ with $h^n = x$. Applying φ and using the homomorphism property gives

$$\varphi(h)^n = \varphi(h^n) = \varphi(x) = y.$$

Thus every element of H has an n-th root, so H is divisible.

Conversely, if H is divisible then the same argument applied to $\varphi^{-1} : H \to G$ shows G is divisible. Therefore divisibility is preserved under group isomorphisms. □

Theorem 9.2.3 *(THE HOLOMORPHIC AUTOMORPHISM GROUP ON B^n)*: The holomorphic automorphism group $\operatorname{Aut}(B^n)$ consists solely of biholomorphisms in the form of

$$\boldsymbol{z} \mapsto \boldsymbol{U}^{-1}\boldsymbol{\varphi}_a \circ \boldsymbol{V}\boldsymbol{z}, \tag{9.2.8}$$

where $\boldsymbol{U}, \boldsymbol{V}$ are unitary matrices, $a \in \mathbb{D}$, and $\boldsymbol{\varphi}_a$ is defined as in (9.2.6) (and every such function lies in $\operatorname{Aut}(B^n)$).

Proof: Let $\boldsymbol{f} \in \operatorname{Aut}(B^n)$ be arbitrary, and set $\boldsymbol{\alpha} = \boldsymbol{f}(\boldsymbol{0})$. Then there exists a unitary matrix $\boldsymbol{U}$ such that $\boldsymbol{U}\boldsymbol{\alpha} = (\|\boldsymbol{\alpha}\|, 0, ..., 0)$.

Now let $\boldsymbol{\varphi}_{\|\boldsymbol{\alpha}\|}$ be as in Proposition 9.2.3, mapping $(\|\boldsymbol{\alpha}\|, 0, ..., 0)$ to $\boldsymbol{0}$. Then, the map $\boldsymbol{\varphi}_{\|\boldsymbol{\alpha}\|} \circ \boldsymbol{U}\boldsymbol{f} \in \operatorname{Aut}(B^n)$ fixes $\boldsymbol{0}$, so by Proposition 9.2.4 it is a unitary transformation, say $\boldsymbol{V}$. Therefore,

$$\boldsymbol{\varphi}_{\|\boldsymbol{\alpha}\|} \circ \boldsymbol{U}\boldsymbol{f} \equiv \boldsymbol{V} \Longrightarrow \boldsymbol{f}(\boldsymbol{z}) \equiv \boldsymbol{U}^{-1}\boldsymbol{\varphi}_{\|\boldsymbol{\alpha}\|}^{-1} \circ \boldsymbol{V}\boldsymbol{z}.$$

The converse is trivial. □

9.3 Topological Equivalence and Biholomorphic Equivalence

Lemma 9.3.1: If $\boldsymbol{P}$ is a permutation matrix, and $\boldsymbol{D}$ is a diagonal matrix, then there exists a diagonal matrix $\boldsymbol{D}'$ such that $\boldsymbol{P}\boldsymbol{D} = \boldsymbol{D}'\boldsymbol{P}$. Similarly, there exists a diagonal matrix $\boldsymbol{D}''$ such that $\boldsymbol{D}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{D}''$.

Proof: Let $\boldsymbol{D} = \operatorname{diag}(d_1, ..., d_n)$ and let σ be the permutation corresponding to $\boldsymbol{P}$, $\boldsymbol{P}\boldsymbol{e}_i = \boldsymbol{e}_{\sigma(i)}$ for each standard basis vector $\boldsymbol{e}_i$. Define

$$\boldsymbol{D}' = \operatorname{diag}\Big(d_{\sigma^{-1}(1)}, ..., d_{\sigma^{-1}(n)}\Big).$$

Then for every i,

$$\boldsymbol{P}\boldsymbol{D}\boldsymbol{e}_i = \boldsymbol{P}(d_i\boldsymbol{e}_i) = d_i\boldsymbol{e}_{\sigma(i)},$$

while

$$\boldsymbol{D}'\boldsymbol{P}\boldsymbol{e}_i = \boldsymbol{D}'\boldsymbol{e}_{\sigma(i)} = d_i\boldsymbol{e}_{\sigma(i)}.$$

Hence

$$\boldsymbol{P}\boldsymbol{D}\boldsymbol{e}_i = \boldsymbol{D}'\boldsymbol{P}\boldsymbol{e}_i$$

for all i, so

$$\boldsymbol{P}\boldsymbol{D} = \boldsymbol{D}'\boldsymbol{P}.$$

Now apply this result to $\boldsymbol{P}^\top$, and thus, $(\boldsymbol{P}^\top\boldsymbol{D})^\top = (\boldsymbol{D}''\boldsymbol{P}^\top)^\top \iff \boldsymbol{D}^\top\boldsymbol{P} = \boldsymbol{P}\boldsymbol{D}''^\top \iff \boldsymbol{D}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{D}''$ since diagonal matrices are invariant under transposition. ☐

Theorem 9.3.1 *(Poincaré)*: For any $n \geq 2$, the n-dimensional unit ball B^n and the n-dimensional polydisk $\mathbb{D}^n$ are not biholomorphically equivalent.

Proof: Suppose, for the sake of contradiction, that there exists a biholomorphism $\boldsymbol{\varphi} : \mathbb{D}^n \to B^n$. Let $\boldsymbol{\alpha} = \boldsymbol{\varphi}(\boldsymbol{0}) \in B^n$, and define $\boldsymbol{\Phi} = \boldsymbol{\varphi}_{\|\boldsymbol{\alpha}\|} \circ \boldsymbol{U} \circ \boldsymbol{\varphi}$, where $\boldsymbol{U}$ is a unitary matrix such that $\boldsymbol{U}\boldsymbol{\alpha} = (\|\boldsymbol{\alpha}\|, 0, ..., 0)$ and $\boldsymbol{\varphi}_{\|\boldsymbol{\alpha}\|}$ is as in Proposition 9.2.3.

The definition of $\boldsymbol{\Phi}$ ensures that $\boldsymbol{\Phi} : \mathbb{D}^n \to B^n$ and $\boldsymbol{\Phi}(\boldsymbol{0}) = \boldsymbol{0}$. Then $\boldsymbol{\Phi}^{-1} \circ \operatorname{Aut}(B^n) \circ \boldsymbol{\Phi}$ consists of functions mapping $\mathbb{D}^n$ to $\mathbb{D}^n$, or that

$$\boldsymbol{\Phi}^{-1} \circ \operatorname{Aut}(B^n) \circ \boldsymbol{\Phi} \subseteq \operatorname{Aut}(\mathbb{D}^n) \implies \operatorname{Aut}(B^n) \subseteq \boldsymbol{\Phi} \circ \operatorname{Aut}(\mathbb{D}^n) \circ \boldsymbol{\Phi}^{-1}.$$

Similarly, $\boldsymbol{\Phi} \circ \operatorname{Aut}(\mathbb{D}^n) \circ \boldsymbol{\Phi}^{-1} \subseteq \operatorname{Aut}(B^n)$. Therefore, $\operatorname{Aut}(B^n) = \boldsymbol{\Phi} \circ \operatorname{Aut}(\mathbb{D}^n) \circ \boldsymbol{\Phi}^{-1}$, and

$$\boldsymbol{\psi} \mapsto \boldsymbol{\Phi} \circ \boldsymbol{\psi} \circ \boldsymbol{\Phi}^{-1} \tag{9.3.1}$$

defines a group isomorphism between $\operatorname{Aut}(\mathbb{D}^n)$ and $\operatorname{Aut}(B^n)$. Let $\operatorname{Aut}'(\mathbb{D}^n) < \operatorname{Aut}(\mathbb{D}^n)$ and $\operatorname{Aut}'(B^n) < \operatorname{Aut}(B^n)$ be subgroups fixing $\boldsymbol{0}$. Therefore, (9.3.1) induces a group isomorphism between $\operatorname{Aut}'(\mathbb{D}^n)$ and $\operatorname{Aut}'(B^n)$ as well.

By Theorem 9.2.1, every element of $\operatorname{Aut}'(\mathbb{D}^n)$ may be uniquely identified with a matrix in the form of

$$\boldsymbol{P}\operatorname{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, ..., \mathrm{e}^{\mathrm{i}\theta_n}),$$

where $\boldsymbol{P}$ is a permutation matrix and $(\theta_1, ..., \theta_n) \in [0, 2\pi)^n$. Hence $\operatorname{Aut}'(\mathbb{D}^n)$ is isomorphic to the group of unitary monomial matrices. The structure of $\operatorname{Aut}'(B^n)$ is given by Proposition 9.2.4, and each element corresponds uniquely to a unitary matrix. Thus there is a natural isomorphism $\mathrm{Aut}'(B^n) \cong \mathrm{U}(n)$, the $n \times n$ *unitary group*.

For $\boldsymbol{U} \in \mathrm{U}(n)$, the spectral theorem allows it to be expressed in the form of $\boldsymbol{V} \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\theta_1}, ..., \mathrm{e}^{\mathrm{i}\theta_n}\right)\boldsymbol{V}^\dagger$. Hence, for any positive integer m, we have

$$\boldsymbol{V} \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\frac{\theta_1}{m}}, ..., \mathrm{e}^{\mathrm{i}\frac{\theta_n}{m}}\right)\boldsymbol{V}^\dagger \in \mathrm{U}(n)$$

and

$$\begin{aligned}&\left(\boldsymbol{V} \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\frac{\theta_1}{m}}, ..., \mathrm{e}^{\mathrm{i}\frac{\theta_n}{m}}\right)\boldsymbol{V}^\dagger\right)^m \\ &= \boldsymbol{V} \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\frac{\theta_1}{m}}, ..., \mathrm{e}^{\mathrm{i}\frac{\theta_n}{m}}\right)\boldsymbol{V}^\dagger \cdots \boldsymbol{V} \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\frac{\theta_1}{m}}, ..., \mathrm{e}^{\mathrm{i}\frac{\theta_n}{m}}\right)\boldsymbol{V}^\dagger.\end{aligned}$$

The adjacent products of $\boldsymbol{V}^\dagger\boldsymbol{V}$ simplify to the identity and the entire expression then simplifies to $\boldsymbol{U}$. Hence the unitary group is divisible.

Consider the unitary monomial matrix

$$\boldsymbol{P}_\tau = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

inducing the permutation τ, swapping the first and second entries. Assume that there exists some unitary monomial matrix $\boldsymbol{Q} = \boldsymbol{P}_\sigma\boldsymbol{D}$ (where $\boldsymbol{D}$ is diagonal and $\boldsymbol{P}_\sigma$ is a permutation matrix corresponding to the permutation σ) such that $\boldsymbol{Q}^2 = \boldsymbol{P}_\tau$. This is equivalent to

$$\boldsymbol{P}_\sigma\boldsymbol{D}\boldsymbol{P}_\sigma\boldsymbol{D} = \boldsymbol{P}_\tau \Longrightarrow \boldsymbol{P}_\sigma\boldsymbol{P}_\sigma\boldsymbol{D}'\boldsymbol{D} = \boldsymbol{P}_\tau \Longrightarrow \boldsymbol{P}_\sigma^2\boldsymbol{D}'' = \boldsymbol{P}_\tau,$$

where $\boldsymbol{D}'$ and $\boldsymbol{D}''$ are diagonal matrices, where the former existence are given by Lemma 9.3.1. Thus, $\boldsymbol{P}_\sigma^2 = \boldsymbol{P}_\tau$ (and $\boldsymbol{D}'' = \boldsymbol{I}$) since their permutation parts must match. This is an impossibility since $\boldsymbol{P}_\sigma^2$ corresponds to an even permutation, while $\boldsymbol{P}_\tau$ corresponds to an odd permutation. Thus, the unitary monomial group is not divisible.

By Proposition 9.2.5, the two groups cannot be isomorphic to each other.

This contradicts the existence of (9.3.1), and therefore, no such biholomorphism φ exists. □

Remark: A more succinct proof of the nonexistence of an isomorphism in the proof of Theorem 9.3.1 can be briefly described by means of topology:

Let M_n denote the subgroup of all monomial matrices in $\mathrm{U}(n)$, or the subgroup of unitary matrices with exactly one nonzero entry in each row and each column, and those nonzero entries lying in $\mathrm{U}(1)$.

For each permutation $\sigma \in S_n$ (the *symmetric group* of permutations) let $\boldsymbol{P}_\sigma$ be the corresponding permutation matrix and define

$$T_\sigma = \left\{\boldsymbol{D}\boldsymbol{P}_\sigma : \boldsymbol{D} = \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\theta_1}, ..., \mathrm{e}^{\mathrm{i}\theta_n}\right) \in \mathrm{U}(1)^n\right\}.$$

Each T_σ is homeomorphic to the torus $\mathrm{U}(1)^n$, and every element of M_n lies in exactly one T_σ. Hence

$$M_n = \bigcup_{\sigma \in S_n} T_\sigma,$$

a disjoint union of $|S_n| = n!$ copies of $\mathrm{U}(1)^n$.

Each T_σ is clopen in M_n by their pairwise disjointness, the topology of the torus, and the fact that their union is M_n. Therefore each T_σ is a connected component of M_n. Because each T_σ is connected, M_n has $n!$ connected components.

The elements of $\mathrm{U}(n)$ may be unitarily diagonalized (by the spectral theorem) into $\boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^\dagger$, where

$$\boldsymbol{D} = \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\theta_1}, ..., \mathrm{e}^{\mathrm{i}\theta_n}\right)$$

is a diagonal unitary matrix and $\boldsymbol{V} \in \mathrm{U}(n)$. Then there exists a connected path connecting $(\theta_1, ..., \theta_n)$ to $(0, ..., 0)$, which corresponds to the matrix $\boldsymbol{V}\boldsymbol{I}\boldsymbol{V}^\dagger = \boldsymbol{I}$. Because every matrix is path-connected to the identity, $\mathrm{U}(n)$ is connected.

Consequently for $n \geq 2$, the subgroup M_n (which has more than one connected component) cannot be isomorphic to $\mathrm{U}(n)$ as a topological group.

Although we do not justify these topological claims in detail here, it is worth noting, heuristically, why such topological considerations naturally arise.

A biholomorphism between two domains induces a homeomorphism with respect to their natural topologies (the *compact-open topology*) by $\psi \mapsto \boldsymbol{\Phi} \circ \psi \circ \boldsymbol{\Phi}^{-1}$. Hence any induced topological invariant of an automorphism group, such as connectivity, is to be preserved under this equivalence. Of course, we are yet to verify the rigor and intuition used within the topology, but the intuitive picture already hints to the validity of the connectivity argument.

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