

Underactuated Robotics

MIT OpenCourseWare 6.8210

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Source: <https://github.com/slipperking/underactuated-robotics>

Covers nonlinear dynamics and control of underactuated mechanical systems, with an emphasis on computational methods. Topics include the nonlinear dynamics of robotic manipulators, applied optimal and robust control and motion planning. Discussions include examples from biology and applications to legged locomotion, compliant manipulation, under-water robots, and flying machines.

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Preface

The goals of the course will be to incorporate the following:

- Non-linear dynamics
- Optimization
- Control
- Machine Learning

using various concepts from the following subjects:

- Linear algebra
- Differential Equations
- Python
- Neural Network jargon

1 Preliminaries

2 Robot Dynamics

In the field of controlling and training of autonomous stuff, there are three popular ways

1. reinforcement learning (in quadrupeds)
2. imitation learning (in manipulation robots)
3. mechanics optimization (as in ATLAS)

Definition 2.1.1 (Underactuated and Fully-Actuated). The equation in (†) is said to be *fully actuated* iff $\mathbf{f}_2$ is of full row rank, or if the rank $\mathbf{f} = \dim \mathbf{q}$ (since $\dim \mathbf{q}$ is the number of rows of $\mathbf{f}_2$). If instead $\text{rank } \mathbf{f}_2 < \dim \mathbf{q}$, the system is then said to be *underactuated*.

Definition 2.1. Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem. Ut enim aequaleam animo, cum corpore dolemus, fieri tamen permagna accessio potest, si aliquod aeternum et infinitum impendere malum nobis opinemur. Quod idem licet transferre in voluptatem, ut postea variari voluptas distinguere possit, augeri amplificarique non possit. At.

$$HiPart 2^{\#} \text{ fails} \quad (2.1)$$

1. 2
2. what

Part $2^{\#}$ (2.1) $^{\#}$

2.1 System Modeling

Usually, controlling a system involves using a loop that adjusts the input of the system by observing the state of the system. How exactly the observants affect the input and how to design this relationship to change the system to a desired state is the study of control theory.

The mathematical background consists mainly of nonlinear differential equations and linear algebra.

Mostly, for differential equations, we will consider how the state vector would change given the current state vector and the control input.

Let $\mathbf{x}$ be the state vector, and let $\mathbf{u}$ be the set of control inputs. Then this relationship is modelled by some function f of $\mathbf{x}$ and $\mathbf{u}$ with

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}).$$

In the linear case,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}.$$

In many mechanical systems, by virtue of Newton's second law, which states that force changes the second-order derivative of position, $\ddot{\mathbf{q}}$, depends on both the generalized positions $\mathbf{q}$, and the generalized velocities $\dot{\mathbf{q}}$. Therefore,

$$\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{u}),$$

as a second-order system. The state of a system comprises not only the positions, but their velocities. Then we write that

$$\mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}, \quad \text{and} \quad \dot{\mathbf{x}} = \begin{pmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} \dot{\mathbf{q}} \\ \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{u}) \end{pmatrix}.$$

If $\mathbf{u}$ represents the motor torques (for example), and the generalized positions measure angles, then this is typically written as

$$\ddot{\mathbf{q}} = \mathbf{f}_1(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{f}_2(\mathbf{q}, \dot{\mathbf{q}})\mathbf{u}, \quad (\dagger)$$

where $\mathbf{f}_1(\cdot) \in \mathbb{R}^m$ is a vector with dimensionality matching $\mathbf{q} \in \mathbb{R}^m$, and $\mathbf{f}_2$ is a matrix of dimension $m \times n$.

Affinity is perceived with respect to the control input $\mathbf{u} \in \mathbb{R}^n$ (hence *control affine*), and the equation above is affine in $\mathbf{u}$ since it is a linear transformation of it.

$$E = mc^2 \quad (2.1.1)$$

Section 2.1[¶] and (2.1.1)[¶]

Definition 2.1.1 (Underactuated and Fully-Actuated). The equation in $(\dagger)$ is said to be *fully actuated* iff $\mathbf{f}_2$ is of full row rank, or if the rank $\mathbf{f} = \dim \mathbf{q}$ (since $\dim \mathbf{q}$ is the number of rows of $\mathbf{f}_2$). If instead rank $\mathbf{f}_2 < \dim \mathbf{q}$, the system is then said to be *underactuated*.

Obviously, in the case that $\mathbf{f}_2$ has more rows (m) than columns (n), then the rank of $\mathbf{f}_2$ is at most n , whereas the dimensionality of $\mathbf{q}$, or m , is greater than n . Thus, in this case the system will always be underactuated. We provide a heuristic or intuitive explanation; since there are less control inputs than the positions we wish to control, we do not *have full control* of the system. Refer to Definition 2.1.1[¶]

2.1.1 Test 2ie^{iθ}

Usually, controlling a system involves using a loop that adjusts the input of the system by observing the state of the system. How exactly the observants affect the input and how to design this relationship to change the system to a desired state is the study of control theory.

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Affinity is perceived with respect to the control input $\mathbf{u} \in \mathbb{R}^n$ (hence *control affine*), and the equation above is affine in $\mathbf{u}$ since it is a linear transformation of it.

Definition 2.1.1.1 (Much). Ah vous dirai je maman

Proof. Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem.

Ut enim aequaleam animo, cum corpore dolemus, fieri tamen permagna accessio potest, si aliquod aeternum et infinitum impendere.

$$\int \int \int \int \int \int \int \int \int \int \int \int dx \, dx \, dx \, dx \, dx \, dx \, dx \, dx \, dx \, dx \, dx \, dx \quad \square$$

Lorem ipsum.

$$F = ma$$

Lorem ipsum.

$$F = ma \quad (2.1.1.2)$$

Lorem ipsum.

$$F = ma$$

Lorem ipsum.

$$F = ma \quad (2.1.1.3)$$

Solution of a test. Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem. Ut enim aequaleam animo, cum corpore dolemus, fieri tamen permagna accessio potest, si aliquod aeternum et infinitum impendere.

1. Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat.
2. Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat.
3. Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat. $\square$

Obviously, in the case that $\mathbf{f}_2$ has more rows (m) than columns (n), then the rank of $\mathbf{f}_2$ is at most n , whereas the dimensionality of $\mathbf{q}$, or m , is greater than n . Thus, in this case the system will always be underactuated. We provide a heuristic or intuitive explanation; since there are less control inputs than the positions we wish to control, we do not *have full control* of the system. Refer to Definition 2.1.1.1^{##}

Figure 2.1^{##}

Really¹, I'm done with Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem. Ut enim aequaleam animo, cum corpore dolemus, fieri tamen permagna accessio potest, si aliquod aeternum et infinitum impendere. and know that

¹Im done

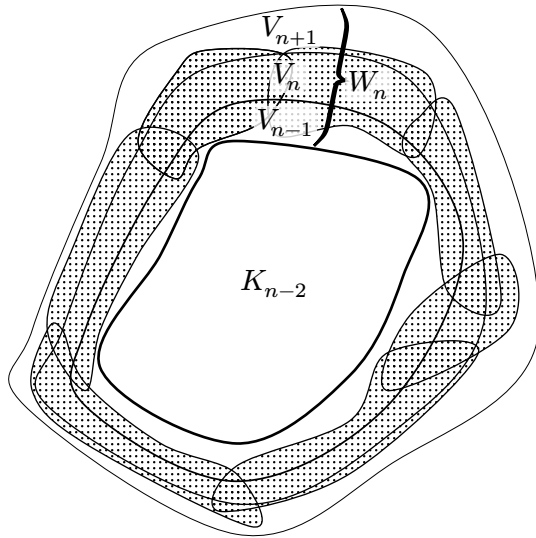


Figure 2.1: Geometry of the finite subcover of $V_n \subset W_n$ for some $n \in \mathbb{N}$.

know that Figure 2.1[‡] is true

2

Proof.

$$\sin \sin \sin \sin \sin \sin \sin \sin \sin \sin \qquad (2.1.1.4)$$

²Im done 2π

$$\begin{aligned}
& \|\sigma(w_1) - \sigma(w_2)\| \\
&= \sqrt{\left(\frac{2\Re w_1}{|w_1|^2+1} - \frac{2\Re w_2}{|w_2|^2+1}\right)^2 + \left(\frac{2\Im w_1}{|w_1|^2+1} - \frac{2\Im w_2}{|w_2|^2+1}\right)^2} \\
&\quad + \sqrt{\left(\frac{|w_1|^2-1}{|w_1|^2+1} - \frac{|w_2|^2-1}{|w_2|^2+1}\right)^2} \\
&= \frac{1}{(|w_1|^2+1)(|w_2|^2+1)} \\
&\quad \times \sqrt{4\Re[w_1(|w_2|^2+1) - w_2(|w_1|^2+1)]^2} \\
&\quad + 4\Im[w_1(|w_2|^2+1) - w_2(|w_1|^2+1)]^2 \\
&\quad + [(|w_1|^2-1)(|w_2|^2+1) - (|w_2|^2-1)(|w_1|^2+1)]^2} \\
&= \frac{2}{\rho_1\rho_2} \sqrt{|w_1\rho_2 - w_2\rho_1|^2 + (\rho_1 - \rho_2)^2} \quad (\text{where } \rho_i = |w_i|^2 + 1) \\
&= \frac{2}{\rho_1\rho_2} \sqrt{\rho_2^2(\rho_1 - 1) + \rho_1^2(\rho_2 - 1) - 2\rho_2\rho_1 \Re[w_1\overline{w_2}] + (\rho_1 - \rho_2)^2} \\
&= \frac{2}{\rho_1\rho_2} \sqrt{\rho_2^2\rho_1 + \rho_1^2\rho_2 - 2\rho_2\rho_1 \Re[w_1\overline{w_2}] - 2\rho_1\rho_2} \quad (\text{by (4.1)}) \\
&= \frac{2}{\sqrt{\rho_1\rho_2}} \sqrt{\rho_2 + \rho_1 - 2\Re[w_1\overline{w_2}] - 2} = \frac{2|w_1 - w_2|}{\sqrt{\rho_1\rho_2}} = k(w_1, w_2) \quad \square
\end{aligned}$$

A Test $\frac{\alpha}{2}$

In the field of controlling and training of autonomous systems, there are three popular methods

1. reinforcement learning (in quadrupeds)
2. imitation learning (in manipulation robots)
3. mechanics optimization (as in ATLAS)

A.1 Gamma β

DUHH

A.1.1 Zeta $\int_0^{2\pi}$

$$F = ma \quad (\text{by (B.1.1)}^\#)$$

A.2 Omicron $\int_0^{2\pi}$

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem. Ut enim aequale doleamus animo, cum corpore dolemus, fieri tamen permagna accessio potest, si aliquod aeternum et infinitum impendere malum nobis opinemur. Quod idem licet transferre in voluptatem, ut postea variari voluptas distinguere possit, augeri amplificarique non possit. At etiam Athenis, ut e patre audiebam facete et urbane Stoicos irridente, statua est in quo a nobis philosophia defensa et collaudata est, cum id, quod maxime placeat, facere possimus, omnis voluptas assumenda est, omnis dolor repellendus. Temporibus autem quibusdam et aut officiis debitis aut rerum necessitatibus saepe eveniet, ut et voluptates repudiandae sint et molestiae non recusandae. Itaque earum rerum defuturum, quas natura non depravata desiderat. Et quem ad me accedis, saluto: 'chaere,' inquam, 'Tite!' lictores, turma omnis chorusque: 'chaere, Tite!' hinc hostis mi Albucius, hinc inimicus. Sed iure Mucius. Ego autem mirari satis non queo unde hoc sit tam insolens domesticarum rerum fastidium. Non est omnino hic docendi locus; sed ita prorsus existimo, neque eum Torquatum, qui hoc primus cognomen invenerit, aut torquem illum hosti detraxisse, ut aliquam ex eo est consecutus? – Laudem et caritatem, quae sunt vitae sine.

$$o \quad (\text{A.2.1})$$

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem. Ut enim aequale doleamus animo, cum corpore dolemus, fieri tamen permagna accessio potest, si aliquod aeternum et infinitum impendere malum nobis opinemur. Quod idem licet transferre in voluptatem, ut postea variari voluptas distinguere possit, augeri amplificarique non possit. At etiam Athenis, ut e

patre audiebam facete et urbane Stoicos irridente, statua est in quo a nobis philosophia defensa et collaudata est, cum id, quod maxime placeat, facere possimus, omnis voluptas assumenda est, omnis dolor repellendus. Temporibus autem quibusdam et aut officiis debitis aut rerum necessitatibus saepe eveniet, ut et voluptates repudiandae sint et molestiae non recusandae. Itaque earum rerum defuturum, quas natura non depravata desiderat. Et quem ad me accedis, saluto: 'chaere,' inquam, 'Tite!' lictores, turma omnis chorusque: 'chaere, Tite!' hinc hostis mi Albucius, hinc inimicus. Sed iure Mucius. Ego autem mirari satis non queo unde hoc sit tam insolens domesticarum rerum fastidium. Non est omnino hic docendi locus; sed ita prorsus existimo, neque eum Torquatum, qui hoc primus cognomen invenerit, aut torquem illum hosti detraxisse, ut aliquam ex eo est consecutus? – Laudem et caritatem, quae sunt vitae sine. (A.2.1)^{##}, (B.1.1)^{##}

Proof. Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem. Ut enim aequae doleamus animo, cum corpore dolemus, fieri tamen permagna accessio potest, si aliquod aeternum et infinitum impendere.

$$F = ma \quad (\text{A.2.2})$$

(A.2.2)^{##} □

Solution to (B.1.1)^{##}. Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem. Ut enim aequae doleamus animo, cum corpore dolemus, fieri tamen permagna accessio potest, si aliquod aeternum et infinitum impendere. □

B List of Theorems

Definition 2.1	1
Definition 2.1.1 (Underactuated and Fully-Actuated)	1
Definition 2.1.1.1 (Much)	1

Bibliography

- [1] R. Tedrake, *Underactuated Robotics*. 2023. [Online]. Available: <https://underactuated.csail.mit.edu/>